throughout his whole career showed a lively and intelligent interest in art and science, and who had taken favorable notice of Leonardo's *Liber Abaci*. In the dedication, dated in 1225, Leonardo relates that he had been presented to the Emperor at court in Pisa, and that Magister Johannes of Palermo had there proposed a problem[1] as a test of Leonardo's mathematical power. The problem was, to find a square number which when either increased or diminished by 5 should still give a square number as result. Leonardo gave a correct answer, $11\frac{97}{144}$. For $11\frac{97}{144} = (3\frac{5}{12})^2$, $6\frac{97}{144} = (2\frac{7}{12})^2$, and $16\frac{97}{144} = (4\frac{1}{12})^2$. Through considering this problem and others allied to it, Leonardo was led to write the *Liber Quadratorum*.[2] It should be said that this problem had been considered by Arab writers with whose works Leonardo was unquestionably familiar; but his methods are original, and our admiration for them is not diminished by careful study of what had been done by his Arabian predecessors.[3]

In the *Liber Quadratorum*, Leonardo has given us a well-arranged, brilliantly-written collection of theorems from indeterminate analysis involving equations of the second degree. Many of the theorems themselves are original, and in the case of many others the proofs are so. The usual method of proof employed is to reason upon general numbers, which Leonardo represents by line segments. He has, it is scarcely necessary to say, no algebraic symbolism, so that each result of a new operation (unless it be a simple addition or subtraction) has to be represented by a new line. But for one who had studied the "geometric algebra" of the Greeks, as Leonardo had, in the form in which the Arabs used it,[4] this method offered some of the advantages of our symbolism; and at any rate it is marvelous with what ease Leonardo keeps in his mind the relation between two lines and with what skill he chooses the right road to bring him to the goal he is seeking.

To give some idea of the contents of this remarkable work, there follows a list of the most important results it contains. The numbering of the propositions is not found in the original.

PROPOSITION I. THEOREM. Every square number[5] can be formed as a sum of successive odd numbers beginning with unity. That is,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

PROPOSITION II. PROBLEM. To find two square numbers whose sum is a square number. "I take any odd square I please, . . . and find the other from

[1] In the introduction to *Flos* we are told that two other problems were propounded at the same time. *Scritti*, II, p. 227.

[2] *Scritti*, II, p. 253.

[3] See, for example, Woepcke, *Recherches sur plusieurs ouvrages de Leonard de Pise, et sur les rapports qui existent entre ces ouvrages et les travaux mathématiques des Arabes*, Rome, 1859.

[4] Heath, T. L., *The Thirteen Books of Euclid's Elements*, Cambridge, 1908. Vol. I, pp. 372–374, 383–385, 386–388; Zeuthen, H. G., *Geschichte der Mathematik im Altertum und Mittelalter*, Copenhagen, 1896, pp. 44–53; Karpinski, L. C., *Robert of Chester's Latin translation of the Algebra of Al-Khowarizmi*, New York, 1915, pp. 77–89.

[5] Throughout this article, unless otherwise stated, the word "number" is to be understood as meaning "positive integer."

the sum of all the odd numbers from unity up to that odd square itself."[1] Thus, if $2n+1$ is a square ($=x^2$) then

$$1+3+5+\cdots+(2n-1)+x^2 = n^2+(2n+1)$$
$$= \text{a sum of two squares} = (n+1)^2$$

This is equivalent to Pythagoras's rule for obtaining rational right triangles, as stated by Proclus,[2] viz.,

$$\left(\frac{x^2-1}{2}\right)^2 + x^2 = \left(\frac{x^2+1}{2}\right)^2.$$

For, inasmuch as $2n+1=x^2$, we have $n=\dfrac{x^2-1}{2}$ and $n+1=\dfrac{x^2+1}{2}$.

PROPOSITION III. THEOREM. $\left(\dfrac{n^2}{4}-1\right)^2+n^2=\left(\dfrac{n^2}{4}+1\right)^2$.

This enables us to obtain rational right triangles in which the hypotenuse exceeds one of the legs by 2. It is attributed by Proclus to Plato[2]. Leonardo also gives the rule in case the hypotenuse is to exceed one leg by 3, and indicates what the result would be if the hypotenuse exceeds one leg by any number whatever.

PROPOSITION IV. THEOREM. "Any square exceeds the square which immediately precedes it by the amount of the sum of their roots." That is, $n^2-(n-1)^2=n+(n-1)$. It follows from this that when the sum of two consecutive numbers is a square number, then the square of the greater will equal the sum of two squares. For, if $n+(n-1)=u^2$, then $n^2-(n-1)^2=u^2$ or $n^2=u^2+(n-1)^2$.

PROPOSITION V. PROBLEM. Given $a^2+b^2=c^2$, to find two integral or fractional numbers x, y, such that $x^2+y^2=c^2$. Solution: Find two other numbers m and n such that[3] $m^2+n^2=q^2$. If $q^2 \neq c^2$, multiply the preceding equation by c^2/q^2, obtaining

$$\left(\frac{c}{q}\cdot m\right)^2+\left(\frac{c}{q}\cdot n\right)^2=c^2$$

so that $x=c/q\cdot m$, $y=c/q\cdot n$ is a solution.

PROPOSITION VI. THEOREM. "If four numbers not in proportion are given, the first being less than the second, and the third less than the fourth, and if the sum of the squares of the first and second is multiplied by the sum of the squares of the third and fourth, there will result a number which will be equal in two ways to the sum of two square numbers." That is,

$$(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2=(ad+bc)^2+(ac-bd)^2.$$

This very important theorem should be called Leonardo's Theorem, for it is

[1] The use of quotation marks indicates a literal translation of Leonardo's words; in other cases the exposition follows his thought without adhering closely to his form of expression.

[2] Proclus, ed. Friedlein, Leipzig, 1873, p. 428.

[3] This is possible by Proposition II or Proposition III.

not found definitely stated, to say nothing of being proved, in any earlier work. Leonardo considers also the case where a, b, c, and d are in proportion, and shows that then $(a^2 + b^2)\cdot(c^2 + d^2)$ is equal to a square and the sum of two squares. This gives him still another way of finding rational right triangles.[1]

PROPOSITION VII. THEOREM. $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$.[2] Leonardo proves this very simply as a corollary of Proposition VI.

PROPOSITION VIII. PROBLEM. "To find two numbers the sum of whose squares is a number, not a square, formed from the addition of two given squares." That is, to find x and y such that $x^2 + y^2 = a^2 + b^2$. Choose any two numbers c and d, such that $c^2 + d^2$ is a square, and write $(a^2 + b^2)(c^2 + d^2)$ as a sum of two squares, let us say $p^2 + q^2$; this we can do by Proposition VI. Construct the right triangle whose legs are p and q; then the similar triangle whose hypotenuse is equal to $\sqrt{c^2 + d^2}$ will have as its legs the two required numbers x and y.

PROPOSITION IX. THEOREM.

$$6(1^2 + 2^2 + 3^2 + \cdots + n^2) = n(n + 1)(2n + 1).$$

The proof of this is strikingly original, and proceeds from the identity

$$n(n + 1)(2n + 1) = n(n - 1)(2n - 1) + 6n^2.$$

Hence

$$n(n - 1)(2n - 1) = (n - 1)(n - 2)(2n - 3) + 6(n - 1)^2,$$

. . . . . .

$$2\cdot3\cdot(2 + 3) = 1\cdot2\cdot(1 + 2) + 6\cdot2^2,$$

$$1\cdot2\cdot(1 + 2) = 6\cdot1^2.$$

It follows by addition that

$$n(n + 1)(2n + 1) = 6(1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2).$$

PROPOSITION X. THEOREM.

$$12[1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2] = (2n - 1)(2n + 1)4n.$$

Leonardo gives a proof very similar to that of Proposition IX.

PROPOSITION XI. THEOREM.

$$12[2^2 + 4^2 + 6^2 + \cdots + (2n)^2] = 2n(2n + 2)(4n + 2),$$

and likewise

$$18[3^2 + 6^2 + 9^2 + \cdots + (3n)^2] = 3n(3n + 3)(6n + 3),$$

and

$$24[4^2 + 8^2 + 12^2 + \cdots + (4n)^2] = 4n(4n + 4)(8n + 4),$$

[1] For instance, letting $a = 6$, $b = 4$, $c = 3$, $d = 2$,

$$(36 + 16)(9 + 4) = 676 = (6\cdot3 + 4\cdot2)^2 = (6\cdot2 + 4\cdot3)^2 + (6\cdot3 - 4\cdot2)^2 = 26^2 = 24^2 + 10^2.$$

[2] This is Euclid's general solution of the problem of finding rational right triangles. Heath, *op. cit.*, III, p. 63. (Euclid's Elements, X, Lemma to Theorem 29.)

and in general

$$6a[a^2 + (2a)^2 + (3a)^2 + \cdots + (na)^2] = na(na + a)(2na + a).$$

Here Leonardo has almost discovered the general result

$$a^2 + (a + d)^2 + (a + 2d)^2 + \cdots + [a + (n - 1)d]^2 = \frac{6na^2 + 6n(n - 1)ad + n(n - 1)(2n - 1)d^2}{6}.$$

His method needed no change at all, in fact.

PROPOSITION XII. THEOREM. If $x + y$ is even, $xy(x + y)(x - y)$ is divisible by 24; and in any case $4xy(x + y)(x - y)$ is divisible by 24. A number of this form is called by Leonardo a *congruum*, and he proceeds to show that it furnishes the solution to a problem proposed by Johannes of Palermo.

PROPOSITION XIII. PROBLEM. "To find a number which, being added to, or subtracted from, a square number, leaves in either case a square number." Leonardo's solution of this, the problem which had stimulated him to write the *Liber Quadratorum*, is so very ingenious and original that it is a matter of regret that its length prevents its inclusion here. It is not too much to say that this is the finest piece of reasoning in number theory of which we have any record, before the time of Fermat. Leonardo obtains his solution by establishing the identities

$$(x^2 + y^2)^2 - 4xy(x^2 - y^2) = (y^2 + 2xy - x^2)^2$$

and

$$(x^2 + y^2)^2 + 4xy(x^2 - y^2) = (x^2 + 2xy - y^2)^2.$$

PROPOSITION XIV. PROBLEM. To find a number of the form $4xy(x+y)(x-y)$ which is divisible by 5, the quotient being a square. Take $x = 5$, and y equal to a square such that $x + y$ and $x - y$ are also squares. The least possible value for y is 4, in which case

$$4xy(x + y)(x - y) = 4 \cdot 5 \cdot 4 \cdot 9 \cdot 1 = 720.$$

PROPOSITION XV. PROBLEM. "To find a square number which, being increased or diminished by 5, gives a square number. Let a congruum be taken whose fifth part is a square, such as 720, whose fifth part is 144; divide by this the squares congruent to 720,[1] the first of which is 961, the second 1681, and the third 2401. The root of the first square is 31, of the second is 41, and of the third is 49. Thus there results for the first square $6\frac{97}{144}$, whose root is $2\frac{7}{12}$, which results from the division of 31 by the root of 144, that is, by 12; and for the second, that is, for the required square, there will result $11\frac{97}{144}$, whose root is $3\frac{5}{12}$, which results from the division of 41 by 12; and for the last square there will result $16\frac{97}{144}$, whose root is $4\frac{1}{12}$."

[1] That is, the three squares in arithmetic progression, whose common difference is the congruum 720. They are obtained by Proposition XIII, thus: Taking $x = 5$ and $y = 4$, $y^2 + 2xy - x^2 = 31$, the root of the first square; $x^2 + y^2 = 41$, the root of the second square; and $x^2 + 2xy - y^2 = 49$, the root of the third square.

PROPOSITION XVI. THEOREM. When $x > y$, $(x+y)/(x-y) \neq x/y$. It follows that $x(x-y)$ is not equal to $y(x+y)$, and "from this," Leonardo says, "it may be shown that no square number can be a congruum." For if $xy(x+y)(x-y)$ could be a square, either $x(x-y)$ must be equal to $y(x+y)$, which this proposition proves to be impossible, or else the four factors must severally be squares, which is also impossible. Leonardo to be sure overlooked the necessity of proving this last assertion, which remained unproved until the time of Fermat.[1]

PROPOSITION XVII. PROBLEM. To solve in rational numbers the pair of equations

$$x^2 + x = u^2,$$
$$x^2 - x = v^2.$$

The solution is obtained by means of any set of three squares in arithmetic progression, that is, by means of Proposition XIII. Let us take x_1^2, x_2^2, and x_3^2 for the three squares, and let the common difference, that is, the congruum, be d. Leonardo says that the solution of the problem is obtained by giving x the value x_2^2/d. For then

$$x^2 + x = \frac{x_2^4}{d^2} + \frac{x_2^2}{d} = \frac{x_2^2(x_2^2+d)}{d^2} = \frac{x_2^2 x_3^2}{d^2};$$

and

$$x^2 - x = \frac{x_2^4}{d^2} - \frac{x_2^2}{d} = \frac{x_2^2(x_2^2-d)}{d^2} = \frac{x_2^2 x_1^2}{d^2}.^2$$

PROPOSITION XVIII. PROBLEM. To solve in rational numbers the pair of equations

$$x^2 + 2x = u^2,$$
$$x^2 - 2x = v^2.$$

The method is similar to that in Proposition XVII, the value of x being found to be $2x_2^2/d$. Leonardo adds, "You will understand how the result can be obtained in the same way if three or more times the root is to be added or subtracted."

PROPOSITION XIX. PROBLEM. To solve (in integers) the pair of equations

$$x^2 + y^2 = u^2,$$
$$x^2 + y^2 + z^2 = v^2.$$

Take for x and y any two numbers that are prime to each other and such that the

[1] Fermat, *Oeuvres*, Paris, 1891, vol. 1, p. 340; Heath, *Diophantus of Alexandria*, Cambridge, 1910, p. 293.

[2] The simplest numerical example would be $x_1^2 = 1$, $x_2^2 = 25$, $x_3^2 = 49$, and this is the illustration given by Leonardo. It leads to $x = \frac{5}{24}$, from which we have $x^2 + x = \frac{1 \cdot 2 \cdot 5 \cdot 5}{5 \cdot 7 \cdot 6} = (\frac{5}{24})^2$ and $x^2 - x = \frac{5 \cdot 5}{5 \cdot 7 \cdot 6} = (\frac{5}{24})^2$.

sum of their squares is a square, let us say u^2. Adding all the odd numbers from unity to $u^2 - 2$[1], the result is $((u^2 - 1)/2)^2$.

Now

$$\left(\frac{u^2-1}{2}\right)^2 + u^2 = \left(\frac{u^2+1}{2}\right)^2.$$

Thus

$$z^2 = \left(\frac{u^2-1}{2}\right)^2,$$

and

$$v^2 = \left(\frac{u^2+1}{2}\right)$$

PROPOSITION XX. PROBLEM. To solve in rational numbers the set of equations

$$x + y + z + x^2 = u^2,$$
$$x + y + z + x^2 + y^2 = v^2,$$
$$x + y + z + x^2 + y^2 + z^2 = w^2.$$

By an extension of the method used in Proposition XIX Leonardo obtains the results $x = 3\frac{1}{5}$, $y = 9\frac{3}{5}$, $z = 28\frac{4}{5}$. He even goes farther and obtains the integral solutions $x = 35$, $y = 144$, $z = 360$. He continues, "And not only can three numbers be found in many ways by this method but also four can be found by means of four square numbers, two of which in order, or three, or all four added together make a square number. . . . I found these four numbers, the first of which is 1295, the second $4566\frac{6}{7}$, the third $11417\frac{1}{7}$, and the fourth 79920." In the midst of the explanation of how these values were obtained, the MS. of the *Liber Quadratorum* breaks off abruptly. It is probable, however, that the original work included little more than what the one known Ms. gives. At all events, considering both the originality and power of his methods, and the importance of his results, we are abundantly justified in ranking Leonardo of Pisa as the greatest genius in the field of number theory who appeared between the time of Diophantus and that of Fermat.

[1] Here u^2 is odd, because it is the sum of the squares of two numbers x and y which are prime to each other. It is not possible that both x and y are odd, since $(2m + 1)^2 + (2n + 1)^2 = 4m^2 + 4m + 4n^2 + 4n + 2$, and this is divisible by 2 but not by 4, and hence can not be a square. Thus, of the numbers x and y, one must be even and the other odd, hence $x^2 + y^2$ is odd.

ON THE FORM OF THE POWER SERIES FOR AN ALGEBRAIC FUNCTION.

By E. J. WILCZYNSKI, University of Chicago.

It is a very familiar fact that the expansion of a rational function into a power series is characterized by the property that its coefficients are connected by a so-called scale of relation. In other words: if $w = f(z)$ is a rational function of z, and if w can be expanded into a power series of the form

$$w = f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_k z^k + \cdots,$$

there will exist a positive integer n, such that for all values of $k \geqq n$, a relation of the form

$$a_k = A_1 a_{k-1} + A_2 a_{k-2} + \cdots + A_p a_{k-p}, \qquad (k = n, n+1, n+2, \cdots),$$

will hold between $p+1$ consecutive coefficients of the expansion, the quantities $A_1, A_2, \cdots, A_p$ being constants independent of k. Conversely any expansion of this form defines w as a rational function of z.

Thus, the existence of a scale of relation is a necessary and sufficient condition for a power-series which defines a rational function. The corresponding condition for the power series expansion of a general algebraic function seems to have escaped notice, although it may be obtained by the following simple and elementary process.

Let us consider the algebraic function w, of z, which is defined by the algebraic equation

$$F(z, w) = \varphi_0(z) w^n + \varphi_1(z) w^{n-1} + \cdots + \varphi_{n-1}(z) w + \varphi_n(z) = 0, \tag{1}$$

where the coefficients $\varphi_k(z)$ are polynomials of degree no higher than m, so that we may write

$$\varphi_k(z) = A_0^{(k)} + A_1^{(k)} z + \cdots + A_m^{(k)} z^m, \tag{2}$$

where the quantities $A_i^{(k)}$ are constants.

Let us suppose that one of the roots of (1) may be expanded as a power series in the neighborhood of $z = 0$, and let

$$w = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{i=0}^{\infty} a_i z^i \tag{3}$$

be this power series. From (3) we may obtain the expansions of $w^2, w^3, \cdots, w^n$ by the familiar process of multiplication of power-series. Let

$$w^k = \sum_{i=0}^{\infty} a_i^{(k)} z^i, \qquad (k = 0, 1, 2, \cdots, n), \tag{4}$$

be the expansions obtained in this way.

If (3) actually represents a root of (1), substitution of (4) in (1) must give rise

to an identity; that is, the total coefficient of each power of z must be equal to zero. Let us make this substitution, and equate to zero the coefficient of z^k. For all values of $k \leqq m$, we find

$$
\begin{aligned}
&(A_0^{(0)}a_k^{(n)} + \cdots + A_m^{(0)}a_{k-m}^{(n)}) + (A_0^{(1)}a_k^{(n-1)} + \cdots + A_m^{(1)}a_{k-m}^{(n-1)}) \\
&\quad + \cdots + (A_0^{(i)}a_k^{(n-i)} + \cdots + A_m^{(i)}a_{k-m}^{(n-i)}) \\
&\quad + \cdots + (A_0^{(n-1)}a_k + \cdots + A_m^{(n-1)}a_{k-m}) \\
&\quad + A_k^{(n)} = 0, \qquad (k \leqq m),
\end{aligned} \tag{5}
$$

and for all higher values of k,

$$
\begin{aligned}
&(A_0^{(0)}a_k^{(n)} + \cdots + A_m^{(0)}a_{k-m}^{(n)}) + (A_0^{(1)}a_k^{(n-1)} + \cdots + A_m^{(1)}a_{k-m}^{(n-1)}) \\
&\quad + \cdots + (A_0^{(i)}a_k^{(n-i)} + \cdots + A_m^{(i)}a_{k-m}^{(n-i)}) \\
&\quad + \cdots + (A_0^{(n-1)}a_k + \cdots + A_m^{(n-1)}a_{k-m}) = 0, \\
&\qquad (k = m+1, m+2, m+3, \cdots).
\end{aligned} \tag{6}
$$

If $n = 1$, equation (1) defines w as a rational function of z, and the relations (6) reduce to the familiar scale of relation. In the general case, relation (6) replaces the scale of relation. As in the scale of relation, the coefficients $A_i^{(j)}$ which occur in (6) are independent of k. In fact they are merely the constant coefficients which appear in the defining algebraic equation (1). But the relation (6) is *not* in general, a *linear* relation between the coefficients of the power series. It *is* linear in the $n(m+1)$ quantities $a_k, \cdots a_{k-m}, a_k^{(2)}, \cdots, a_{k-m}^{(2)}, \cdots, a_k^{(n)}, \cdots, a_{k-m}^{(n)}$, but it is of degree n with respect to the quantities $a_1, \cdots, a_k$. We shall therefore speak of a relation of form (6) as *a scale of relation of degree n*. Such a scale is of the same form as the familiar simplest case except that it is concerned not only with the coefficients of the expansion of w itself, but also with the coefficients of the expansion of $w^2, w^3, \cdots, w^n$. We may summarize our result as follows, including an obvious generalization which consists in replacing z by $z - \alpha$.

If w is an algebraic function of z, defined by an equation of degree n in w and of degree m in z, and if w can be expanded in a convergent series proceeding according to positive integral powers of $z - \alpha$, then the coefficients of this expansion satisfy a scale of relation of degree n, which will contain at most $(m+1)n$ terms.

The converse is immediate. If the coefficients a_k of a power series satisfy a given scale of relation of degree n, the coefficients, $A_i^{(j)}$, $(j = 0, 1, \cdots, n-1;$ $i = 0, 1, \cdots, m)$, of the scale will determine the coefficients $\varphi_0(z), \varphi_1(z), \cdots,$ $\varphi_{n-1}(z)$ of an algebraic equation of form (1) between w and z The constant coefficients of $\varphi_n(z)$ may then be determined by means of (5). These latter coefficients depend not only upon the given scale of relation but also upon the values of the first $m+1$ coefficients of the given power series. The algebraic equation between w and z, obtained in this way, will have as one of its roots the function with the given expansion.

Thus, *any convergent power-series, whose coefficients satisfy a scale of relation of degree n, defines an algebraic function which is a root of an equation of degree n in w, and whose coefficients are integral rational functions of z.*

Of course we are concerned only with the form of the power series expansion for w whenever such an expansion *exists*. We have not discussed the exceptional cases when such an expansion is impossible. But these exceptional cases are well known. Thus if α is a zero of $\varphi_0(z)$ the expansion of w in powers of $z - \alpha$ may contain negative powers. If α is a root of the equation obtained by eliminating w from $F(z, w) = 0$ and $\partial F/\partial w = 0$, there may be fractional powers of $z - \alpha$ in such an expansion. If $z = \alpha$ is a value of z different from these critical values, an expansion in powers of $z - \alpha$ involving only positive integral powers is known to exist, and the coefficients of the expansion will obey the law which we have indicated. As a matter of fact such expansions frequently exist even in the neighborhood of the critical values. Moreover it is easy to make the modifications necessary to describe the form of the expansion when fractional or negative powers occur in it. Finally it may be stated that the critical values of z might also be rediscovered by investigating directly those cases in which the series obtained from (5) and (6) fail to converge.

The equation (1) may be reducible, that is, it may be possible to express its left member as a product of two integral rational functions of lower degree. In that case we shall still have a scale of relation of form (6) but with smaller values for m or n or both.

The scale of relation for the coefficients of the power-series expansion of an algebraic function will actually be a scale of degree n (and not lower), if the equation (1) *which defines this algebraic function is an irreducible equation of degree n in w.*

We have found a necessary and sufficient condition in order that the function w defined by the given power-series may be algebraic. We may express this condition in another way.

Let us write the matrix of the coefficients of $A_0^{(0)} \cdots A_m^{(n-1)}$, in equations (6), that is, the matrix of mn columns and infinitely many rows:

$$
\begin{array}{l}
a_{m+1}, \cdots, a_1, a_{m+1}^{(2)}, \cdots, a_1^{(2)}, \cdots, a_{m+1}^{(n)}, \cdots, a_1^{(n)}, \\
a_{m+2}, \cdots, a_2, a_{m+2}^{(2)}, \cdots, a_2^{(2)}, \cdots, a_{m+2}^{(n)}, \cdots, a_2^{(n)}, \\
a_{m+3}, \cdots, a_3, a_{m+3}^{(2)}, \cdots, a_3^{(2)}, \cdots, a_{m+3}^{(n)}, \cdots, a_3^{(n)}, \\
\quad \cdots \cdots \cdots

It is clear that the systematic investigation of all such determinants will disclose the lowest values of m and n which will serve for a given power series, and that we may then actually determine the irreducible algebraic equation which the given power-series will actually satisfy. If no finite values of m and n can be found which satisfy the conditions of the theorem, the given power-series defines a transcendental function.

The remarkable theorem of EISENSTEIN,[1] which furnishes a necessary condition on the coefficients a_k in order that w may be algebraic, in the special case when the coefficients a_k are all rational numbers, may be regarded as a consequence of the general theorems stated in this paper.

CONCERNING A METHOD FOR FINDING A PARTICULAR INTEGRAL.[2]

By ARTHUR B. COBLE, University of Illinois.

The problem considered here is the determination of a particular integral U of the linear differential equation with constant coefficients,

$$(1) \qquad f(D)y \equiv (k_0D^n + k_1D^{n-1} + \cdots + k_n)y = X\left(D = \frac{d}{dx}\right),$$

in the special case when the right-hand member X is the sum of a number of terms $g_1(x), \cdots, g_j(x)$ such that these terms and all of the terms which arise from them by differentiation can be expressed linearly with constant coefficients by means of a finite number

$$(2) \qquad g_1(x), g_2(x), \cdots, g_l(x)$$

of the terms. Such terms for example are x^ρ, e^{ax}, $\sin^\rho \beta x$, $\cos^\rho \gamma x$, $\sinh^\rho \delta x$, $\cosh^\rho \epsilon x$ (ρ any positive integer), and any products of these.

As is customary we call $f(m) = 0$ the *auxiliary equation;* and the complete solution of the differential equation $f(D)y = 0$, the *complementary function* of (1). Each r-fold root, $m = \alpha$, of $f(m) = 0$ contributes a part

$$(3) \qquad G(\alpha, r) = (\lambda_1 x^{r-1} + \lambda_2 x^{r-2} + \cdots + \lambda_r)e^{ax}$$

to the complementary function.

The rule for the determination of the particular integral U of (1) can be stated as follows:

I. *Substitute for y in* (1) *the trial integral* $U = c_1g_1(x) + c_2g_2(x) + \cdots + c_lg_l(x)$ *and determine the coefficients* $c_1, \cdots, c_l$ *by equating coefficients of* $g_1, \cdots, g_l$. *If*

[1] Stated by him without proof. For a proof consult Heine, "Der Eisenstein'sche Satz über Reihenentwickelungen aller algebraischer Funktionen," *Crelle's Journal*, Vol. 45 (1853), pp. 285–382.

[2] Read before the Md.-Va.-D. C. Section of the Mathematical Association, at Annapolis, December 15, 1917.

however X contains a term $g(x)$ or a term $x^{\rho}g(x)$ where $g(x)$ already occurs in the complementary function corresponding to an r-fold root of the auxiliary equation, then in making up the trial integral U, the term $x^{\rho+r}g(x)$ and its derivatives are to be used.[1]

This method for determining U has several advantages. It requires only differentiations and the solution of a system of linear equations. It furnishes an integral expressed in terms of functions similar to those that occur in X. It applies to nearly all types of equations that actually occur in practice. It has the further pedagogic advantage that in any particular example the method justifies itself by furnishing the required integral. The following proof of the range and validity of the method is perhaps of interest only to the teacher or more mature student.

We shall first find an explicit form for the right-hand member X of (1), and second prove that the set described in I of linear equations in $c_1, \cdots, c_l$ is consistent.

By hypothesis the linearly independent functions $g_1(x), \cdots, g_l(x)$ derived from the terms in X and their derivatives are such that

$$
\begin{aligned}
D_x g_1(x) &= g_1' = a_{11}g_1 + a_{12}g_2 + \cdots + a_{1l}g_l, \\
D_x g_2(x) &= g_2' = a_{21}g_1 + a_{22}g_2 + \cdots + a_{2l}g_l, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
D_x g_l(x) &= g_l' = a_{l1}g_1 + a_{l2}g_2 + \cdots + a_{ll}g_l.
\end{aligned} \tag{4}
$$

[2]

If we set

$$g_i = b_1h_1(x) + b_2h_2(x) + \cdots + b_lh_l(x),$$

then

$$g_i' = b_1h_1' + b_2h_2' + \cdots + b_lh_l';$$

whence *the l functions g and the l derivatives g' are cogredient under linear transformation with constant coefficients.* If then the characteristic equation of the linear transformation (4) has roots $\beta_1, \cdots, \beta_k$ of respective multiplicities $s_1, \cdots, s_k$ $(s_1 + \cdots + s_k = l)$ we can find l linearly independent linear functions $h_1(x), \cdots, h_l(x)$ of $g_1(x), \cdots, g_l(x)$ such that the system of equations (4) takes the form[3]

$$
\begin{aligned}
h_1' &= \beta h_1 + h_2 &&&&&&, \\
h_2' &= &&\beta h_2 + h_3 &&&&, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
h_{s-1}' &= &&&&\beta h_{s-1} + h_s &&, \\
h_s' &= &&&&&&\beta h_s,
\end{aligned} \tag{5}
$$

where (5) is a sample of that set of equations of the system which corresponds to a root β of multiplicity s of the characteristic equation.

[1] Cohen's *Differential Equations*, p. 112. So far as the writer is aware the method is first set forth there under the name of "the method of undetermined coefficients."

[2] For a similar system of differential equations cf. Riemann-Weber, *Partielle Differentialgleichungen*, p. 138.

[3] Bôcher's *Higher Algebra*, Chap. XXI.

The solutions of (5) satisfy the system

$$
\begin{aligned}
&h_s' - \beta h_s = 0,\\
&h_{s-1}'' - 2\beta h_{s-1}' + \beta^2 h_{s-1} = 0,\\
(6)\qquad &h_{s-2}''' - 3\beta h_{s-2}'' + 3\beta^2 h_{s-2}' - \beta_3 h_{s-2} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&h_1^{(s)} - \binom{s}{1}\beta h_1^{(s-1)} + \binom{s}{2}\beta^2 h_1^{(s-2)} \cdots + (-1)^s\binom{s}{s}\beta^s h_1 = 0.
\end{aligned}
$$

This is evidently true of the first equation; we have only to verify that it is true of the last when the earlier ones are satisfied. By virtue of the numerical relation

$$\binom{s-1}{a-1} + \binom{s-1}{a} = \binom{s}{a}$$

the last equation can be written

$$
\begin{aligned}
[h_1^{(s)} - \beta h_1^{(s-1)}] - \binom{s-1}{1}\beta[h_1^{(s-1)} - \beta h_1^{(s-2)}] + \binom{s-1}{2}\beta^2[h_1^{(s-2)} - \beta h_1^{(s-3)}]\\
\cdots + (-1)^{s-1}\binom{s-1}{s-1}\beta^{s-1}[h_1' - \beta h_1] = 0.
\end{aligned}
$$

Since $h_1' - \beta h_1 = h_2$ this is satisfied due to the preceding equation in (6). The complete solution of the last equation of (6) is

$$h_1 = e^{\beta x}H(x, s), \qquad H(x, s) = a_1 x^{s-1} + a_2 x^{s-2} + \cdots + a_s$$

and the values of $h_2, \cdots, h_s$ derived in turn from (6) are

$$h_2 = e^{\beta x}D_x H(x, s), \qquad h_3 = e^{\beta x}D_x{}^2 H(x, s), \cdots, h_s = e^{\beta x}D_x{}^{s-1}H(x, s).$$

Thus $h_1, \cdots, h_s$, and therefore the original g's, can be expressed linearly in terms of $x^{s-1}e^{\beta x}, x^{s-2}e^{\beta x}, \cdots, e^{\beta x}$. Hence

II. *The right-hand member X of* (1) *can be expressed as*

$$X = G_1(\beta_1, s_1) + G_2(\beta_2, s_2) + \cdots + G_k(\beta_k, s_k)$$

where the β's are the distinct roots with respective multiplicities s of the characteristic equation of the matrix $|a_{ij}|$ of the system of equations (4) *and the functions G are of the form* (3).

We can obtain a particular integral U, for each part G, of X and by adding the U, get the required integral. If the $X = G(\beta, s)$ we shall take the general case and suppose that β, the s-fold root of the characteristic equation, is also an r-fold root of the auxiliary equation $f(m) = 0$. Then

III. *The method* (4) *furnishes one and only one particular integral of the form*

$$e^{\beta x}[c_1 x^{r+s-1} + c_2 x^{r+s-2} + \cdots + c_s x^r].$$

For if in $f(D)y$ we set

$$y = e^{\beta x}(b_1 x^{t-1} + b_2 x^{t-2} + \cdots + b_{t-1}x + b_t)$$

then the coefficients of $e^{\beta z}x^{t-1}$, $e^{\beta z}x^{t-2}$, $\cdots$, $e^{\beta z}$ in the result are respectively

$$
\begin{aligned}
&f(\beta)\cdot b_1,\\
&f(\beta)\cdot b_2+\binom{t-1}{1}f'(\beta)\cdot b_1,\\
&f(\beta)\cdot b_3+\binom{t-2}{1}f'(\beta)\cdot b_2+\binom{t-1}{2}f''(\beta)\cdot b_1,\\
&f(\beta)\cdot b_4+\binom{t-3}{1}f'(\beta)\cdot b_3+\binom{t-2}{2}f''(\beta)\cdot b_2+\binom{t-1}{3}f'''(\beta)\cdot b_1,\\
&\cdots\cdots\cdots

startling conclusions on the historical question relating to the first invention of the calculus. He places his conclusions in italics in the first sentence of his preface, as follows:

"ISAAC BARROW *was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibniz was also in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's book that he purchased in 1673.*"

These claims are far-reaching. Either they are true and should be accepted or they are false and should be rejected.

Before entering upon an examination of the evidence brought forth by Child, it may be of interest to review a similar claim set up for another man as inventor of the calculus who, like Barrow, was active before the time of Newton and Leibniz. Such a claim has been made in favor of Pierre de Fermat (1601–1665) who died when Newton was just beginning his great career and when Leibniz had not yet brought out his first mathematical publication. Fermat was declared to be the first inventor of the calculus by Lagrange, Laplace, and apparently also by P. Tannery, than whom no more distinguished mathematical triumvirate can easily be found. Lagrange expressed himself as follows:[1]

"One may regard Fermat as the first inventor of the new calculus. In his method *De maximis et minimis* he equates the quantity of which one seeks the maximum or the minimum to the expression of the same quantity in which the unknown is increased by the indeterminate quantity. In this equation he causes the radicals and fractions, if any such there be, to disappear and after having crossed out the terms common to the two numbers, he divides all others by the indeterminate quantity which occurs in them as a factor; then he takes this quantity zero and he has an equation which serves to determine the unknown sought. . . . It is easy to see at first glance that the rule of the differential calculus which consists in equating to zero the differential of the expression of which one seeks a maximum or a minimum, obtained by letting the unknown of that expression vary, gives the same result, because it is the same fundamentally and the terms one neglects as infinitely small in the differential calculus are those which are suppressed as zeroes in the procedure of Fermat. His method of tangents depends on the same principle. In the equation involving the abscissa and ordinate which he calls the specific property of the curve, he augments or diminishes the abscissa by an indeterminate quantity and he regards the new ordinate as belonging both to the curve and to the tangent; this furnishes him with an equation which he treats as that for a case of a maximum or a minimum. . . . Here again one sees the analogy of the method of Fermat with that of the differential calculus; for, the indeterminate quantity by which one augments the abscissa x corresponds to its differential dx, and the quantity ye/t, which is the corresponding augmentation[2] of y, corresponds to the differential dy. It is also remarkable that in the paper which contains the discovery of the differential calculus, printed in the Leipsic Acts of the month of October, 1684, under the title, *Nova methodus pro maximis et minimis* etc., Leibnitz calls dy a line which is to the arbitrary increment dx as the ordinate y is to the subtangent; this brings his analysis and that of Fermat nearer together. One sees therefore that the latter has opened the quarry by an idea that is very original, but somewhat obscure, which consists in introducing in the equation an indeterminate which should be zero by the nature of the question, but which is not made to vanish until after the entire equation has been divided by that same quantity. This idea has become the germ of new calculi which have caused geometry and mechanics to make such progress, but one may say that it has brought also the obscurity of the principles of these calculi. And now that one has a quite clear idea of these principles, one sees

[1] J. Lagrange, "Lecons sur le calcul des fonctions," leçon dix-huitième, *Œuvres de Lagrange*, publiées par J. A. Serret, Tome X, p. 294.

[2] Fermat lets e be the increment of x, and t the subtangent for the point x, y on the curve.

that the indeterminate quantity which Fermat added to the unknown simply serves to form the *derived function* which must be zero in the case of a maximum or minimum, and which serves in general to determine the position of tangents of curves. But the geometers contemporary with Fermat did not seize the spirit of this new kind of calculus; they did not regard it but a special artifice, applicable simply to certain cases and subject to many difficulties, . . . moreover, this invention which appeared a little before the *Géométrie* of Descartes remained sterile during nearly forty years. . . . Finally Barrow contrived to substitute for the quantities which were supposed to be zero according to Fermat quantities that were real but infinitely small, and he published in 1674 his *method of tangents*, which is nothing but a construction of the method of Fermat by means of the infinitely small triangle, formed by the increments of the abscissa e, the ordinate ey/t, and of the infinitely small arc of the curve regarded as a polygon. This contributed to the creation of the system of infinitesimals and of the differential calculus."

Such is Lagrange's interpretation of the work of Fermat and of the place it should occupy in the history of the calculus. Even more positive is the dictum of Pierre Simon Laplace who, in his *Essai philosophique sur le calcul des probabilities*, speaks of Fermat in the following terms:[1]

"This great geometrician [Fermat] expresses by the character E the increment of the abscissa; and considering only the first power of this increment, he determines exactly as we do by differential calculus the subtangents of the curves, their points of inflection, the *maxima* and *minima* of their ordinates, and in general those of rational functions. We see likewise by his beautiful solution of the problem of the refraction of light inserted in the *Collection of the Letters of Descartes* that he knows how to extend his methods to irrational functions in freeing them from irrationalities by the elevation of the roots to powers. Fermat should be regarded, then, as the true discoverer of Differential Calculus. Newton has since rendered this calculus more analytical in his *Method of Fluxions*, and simplified and generalized the processes by his beautiful theorem of the binomial. Finally, about the same time Leibnitz has enriched differential calculus by a notation which, by indicating the passage from the finite to the infinitely small, adds to the advantage of expressing the general results of calculus, that of giving the first approximate values of the differences and of the sums of the quantities; this notation is adapted of itself to the calculus of partial differentials."

P. Tannery, the noted historian of mathematics, expressed himself more recently as follows:[2]

"Fermat is also honored with the invention of the differential calculus on account of his method of maxima and minima and of tangents, which, of the prior processes, is in reality the nearest to the algorithm of Leibniz; one could with equal justice, attribute to him the invention of the integral calculus; his treatise *De æquationum localium transmutatione*, etc., gives indeed the method of integration by parts as well as rules of integration, except the general powers of variables, their sines and powers thereof. However, it must be remarked that one does not find in his writings a single word on the main point, the relation between the two branches of the infinitesimal calculus."

We proceed to quote two opinions, one English, the other French, relating to the attitude taken by Lagrange and Laplace. In the *Edinburgh Review* for September, 1814, p. 324, we read (in an anonymous review now known to have been from the pen of John Playfair):

"To a passage of the latter [Laplace], however, we cannot but advert, and with much less satisfaction than we have generally felt in pointing out any of the remarks of this celebrated writer to the attention of our readers. '*Il parait que Fermat, le veritable inventeur du calcul differentiel,*

[1] *A Philosophical Essay on Probabilities*, by Pierre Simon, Marquis de Laplace. Transl. by F. W. Truscott and F. L. Emory, New York, 1902, Part I, Chapter V, p. 46. The French original appeared in 1812.

[2] P. Tannery, Article "Fermat" in *La Grande Encyclopédie* (Berthelot).

a consideré ce calcul comme une dérivation de celui des differences finies,' etc. Against the affirmation that FERMAT is the real inventor of the Differential Calculus, we must enter a strong and solemn protestation. The age in which that discovery was made, has been unanimous in ascribing the honour of it either to NEWTON or LEIBNITZ; or, as seems to us much the fairest and most probable opinion, to both; that is, to each independently of the other, the priority in respect of time being somewhat on the side of the English mathematician. The writers of the history of the mathematical sciences have given their suffrages to the same effect;—MONTUCLA, for instance, who has treated the subject with great impartiality, and BOSSUT, with no prejudices certainly in favour of the English philosopher. In the great controversy, to which this invention gave rise, all the claims were likely to be well considered; and the ultimate and fair decision, in which all sides seem to have acquiesced, is that which has just been mentioned. It ought to be on good grounds, that a decision, passed by such competent judges, and that has now been in force for a hundred years, should all at once be reversed.—Fermat . . . had certainly approached very near to the differential or fluxionary calculus, as his friend ROBERVAL had also done. He considered the infinitely small quantities introduced in his method of drawing tangents, and of resolving *maxima* and *minima*, as derived from finite differences; and, as Laplace remarks, he has extended his method to a case, when the variable quantity is irrational. He was, therefore, very near to the method of fluxions; with the principle of it, he was perfectly acquainted;—and so at the same time were both ROBERVAL and WALLIS, though men much inferior to FERMAT. The truth is, that the discovery of the new calculus was so gradually approximated, that more than one had come quite near it, and were perfectly acquainted with its principles, before any of the writings of NEWTON or LEIBNITZ were known. That which must give, in such a case, the right of being considered as the true inventor, is the extension of the principle to its full range; connecting with it a new calculus, and new analytical operations; the invention of a new algorithm with corresponding symbols. These last form the public acts, by which the invention becomes known to the world at large the judge by which the matter must be finally decided. Great, therefore, as is the merit of FERMAT which no body can be more willing than ourselves to acknowledge; and near as he was to the greatest invention of modern times, we cannot admit that his property in it is to be put on a footing with that of NEWTON or of LEIBNITZ;—we should fear, that in doing so, we were violating one of the most sacred and august monuments that posterity ever raised in honour of the dead."

Poisson says:[1]

"As a magnitude approaches its maximum or its minimum, it varies less and less and its differential vanishes as it reaches one or the other of these extreme values. Starting from this principle, Fermat had the happy idea, for the determination of the maximum or minimum of a quantity, to assign to the variable upon which it depends, an infinitely small increment and to equate to zero the corresponding increment of that quantity previously reduced to the same order of magnitude as that of the variable. It is in this manner that he determined the path of light in passing from one medium into another upon the supposition conforming with the theory he had adopted, that the time of passage be a minimum. Lagrange considered him, for that reason, as the first inventor of the differential calculus, but this calculus consists of a set of rules suitable for finding immediately the differences of all functions, rather than of the use one makes of the infinitely small variations in the resolution of this or that species of problems; and from that point of view, the creation of the differential calculus does not go back beyond Leibnitz, the inventor of the algorithm and of the notation which have generally prevailed since the origin of the calculus and to which infinitesimal analysis is chiefly indebted for its progress. It should be observed, moreover, that the binomial formula which supplied Newton and Leibnitz the means of expressing very simply the differential of any power whatever, integral or fractional, positive or negative—that this formula was unknown to Fermat, that he could not differentiate the radicals which presented themselves in his problem, and that he replaced this operation by geometrical constructions and special devices, the avoidance of which is the special object of the differential calculus."

We have now quoted the views of Lagrange, Laplace, Poisson, Paul Tannery and an Edinburgh reviewer, on the invention of the calculus. We have quoted

[1] Poisson "Mémoire sur le calcul des Variations," *Mémoires de l'académie royale des sciences de l'institut de France*, Tome XII, Paris, 1833, p. 223.

the conclusion reached by Child in favor of Barrow. The practical question arises, which of these conflicting opinions is correct. It is easy to see that the answer to be given to this question hinges upon the conception we have as to what constitutes an invention of the calculus. Is it the creation of a method like that of Fermat? Is it the invention of a set of rules and a notation as demanded by Poisson?

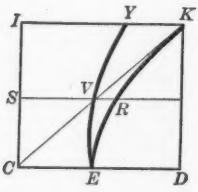
Mr. Child endeavors to answer this question in his preface: "By the 'Infinitesimal Calculus,' I intend 'a complete set of standard forms for both the differential and integral sections of the subject, together with rules for their combination, such as for a product, a quotient, or a power of a function; and also a recognition and demonstration of the fact that differentiation and integration are inverse operations.'"

Every one will admit that a set of rules for differentiation and integration is an essential part of the calculus. But does this include all the essentials to be met by a successful candidate for the honor of invention of the calculus? Could the calculus have fulfilled its mission had it not possessed a suitable notation both for differentiation and integration? We do not mean that the very symbols introduced by Leibniz or by Newton are essential parts of a differential and integral calculus. But is not some sort of a suitable notation to designate the first differential or derivative as well as higher differentials or derivatives and to designate integration a necessary part of the calculus—a *sine qua non*, without which the calculus could not render its service in the resolution of complicated problems? We hold that by common and tacit agreement of all text book writers since the time of Newton and Leibniz such a notation is looked upon as an essential of the calculus. We cannot recall a single author of a text on this subject since the time of Newton and Leibniz who has not recognized the need of a suitable notation and who does not use one. On the strength of this unanimity of tacit testimony we must insist upon a suitable notation as a necessary requirement to be met by any one for whom the invention of the calculus is claimed. In his geometrical lectures Barrow uses few symbols. He does not even use the ordinary symbols in trigonometry. He uses the letters a and e for the designation of increments of variables, but this notation is altogether inadequate.

Passing to another topic, we observe that most texts have invoked the aid of geometry in the development of the calculus. Geometrical figures help in the grasp of abstract relations. Nevertheless, the calculus has been largely analytical. Algebraic, logarithmic and trigonometric symbols have been used habitually. We could not have rules for differentiation and integration, as ordinarily understood, unless the mathematical relations were expressed in analytical form. Now Barrow does not develop an *analytical* calculus. He establishes geometrical theorems which Child translates into analysis. In that way Child obtains a group of formulas for differentiation and integration, such as have been gathered also from other pre-Newtonian writers.[1] These formulas he sets down to Bar-

[1] See H. G. Zeuthen, *Geschichte der Mathematik im XVI. und XVII. Jahrhundert*, deutsche Ausg. von R. Meyer, Leipzig, 1903, "Integrationen vor der Integralrechnung," pp. 248–300.

row's credit. For example Barrow proves the theorem (Lecture XII, Appendix I, 9):



"Let *ERK* be an equilateral hyperbola (that is, one having equal axes), and let the axes be *CED*, *CI*; also let *KI*, *KD* be ordinates to these; let *EVY* be a curve such that, when any point *R* is taken at random on the hyperbola, and a straight line *RVS* is drawn parallel to *DC*, then *SR CE*, *SV* are in continued proportion; join *CK*; then the space *CEYI* will be double the hyperbolic sector *KCE*."

The adjoining figure is our own; Child gives none. The determination of the area *CEYI* by calculus involves the formula

$$\int_0^y \frac{dy}{\sqrt{a^2+y^2}} = \log \frac{y+\sqrt{a^2+y^2}}{a}.$$

Hence Child claims that Barrow possessed this formula in geometrical garb. Accordingly, the geometrical process of integrating

$$\int_0^y \frac{dy}{\sqrt{a^2+y^2}}$$

would be to construct the hyperbola $x^2 - y^2 = a^2$ and from it the curve *EVY*, yielding *CEYI* as the area representing the definite integral. By the same argument it may be claimed that when Dinostratus of old used the quadratrix in the quadrature of the circle, he worked out the part of the integral calculus contained in the formula $\int_0^a \sqrt{a^2 - x^2}dx = \frac{1}{4}\pi a^2$. Dinostratus and Barrow were clever men, but it seems to us that they did not create what by common agreement of mathematicians has been designated by the term differential and integral calculus. Two processes yielding equivalent results are not necessarily the same. It appears to us that what can be said of Barrow is that he worked out a set of geometric theorems suggesting to us constructions by which we can find lines, areas and volumes whose magnitudes are ordinarily found by the analytical processes of the calculus. But to say that Barrow invented a differential and integral calculus is to do violence to the habit of mathematical thought and expression of over two centuries. The invention rightly belongs to Newton and Leibniz.

FLORIAN CAJORI.

UNIVERSITY OF CALIFORNIA.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

The discussion given below is closely related to Professor Moritz's article "On the Construction of Certain Curves Given in Polar Coördinates," published

in the May, 1917, issue of this MONTHLY, pp. 213–220. Professor Moritz discussed "cyclic-harmonic" curves resulting from the motion of a point which has simple harmonic motion along a line, while at the same time this line rotates with a constant angular velocity about one of its fixed points. Professor Rigge gives below a very complete discussion of a method for tracing a cardioid as the resultant of a simple harmonic motion along a line and a uniform angular motion about a point not in that line.

CONCERNING A NEW METHOD OF TRACING CARDIOIDS.

By WILLIAM F. RIGGE, S.J., Creighton University, Omaha, Nebr.

A cardioid may under certain conditions be traced by a point when its motion is the resultant of a rectilinear simple harmonic movement and a uniform angular one of the same period. The harmonic motion may be obtained from any plane mechanical contrivance such as a revolving crank and a sliding slotted bar, while the angular one is best furnished by a disk rotating under the pen.

Fig. 1 will illustrate the definition just given as well as the conditions to be mentioned presently. The point A is the center of the disk which rotates in a clockwise direction with uniform angular speed. If the disk did not rotate, the tracing pen would move over the line EG parallel to the Y axis with simple harmonic motion, so that its distance from R, the middle point, would at any moment be the sine of the phase, the amplitude RE or RG being taken as the unit of our scale in this investigation. But when the disk does rotate, the combination of the rectilinear motion of the pen with the rotary one of the disk causes the pen B to trace the cardioid $BCKQZFB$ (R is only by accident on the curve), provided the following conditions are observed.

Conditions to be Observed.—First, the pen may be set down on the disk as at B at any initial phase α of its rectilinear harmonic motion. In Fig. 1 this initial phase α is taken as 52°, so that $RB = \sin 52°$, RE, as said, being unity.

Second, the point B, at which the pen is set down on the disk, must be on the unit circle whose center O is on the Y axis and whose distance from A, the center of the disk, is the sine of the phase $OA = \sin\alpha = \sin 52° = BR$. The angle AOB we will call β, the *starting* angle, and the circle just mentioned the *starting* circle. In Fig. 1 β is equal to 77°.

Study of the Conditions.—It is to be noted first that α and β are independent variables and may have any values whatever.

Secondly, the initial phase α fixes the center of the starting circle O on the axis of Y, so that O is in the same direction from A that B is from R, and $OA = BR$. It also fixes the center D of what we will call the *cusp*-circle, because it is the locus of the cusp of the cardioid that can be generated with the given initial phase α. The radius of this small circle is one half that of the starting circle, or one half of our chosen unit, its circumference passes through A and O, and it is internally tangent to the large circle at S on the axis of X, so that the angle $ASO = \alpha$. This puts S to the left of A when $\cos\alpha$ is positive, and to the right

when negative. As α grows from 0° to 360° the center O of the starting circle executes a simple harmonic motion along the axis of Y, about its central position at A, while the center D of the cusp circle moves with uniform angular speed in a circle of its own size about the center of the disk A.

Thirdly, the assumed point B at which the pen is set down on the starting circle, or the assumed value of AOB or β, determines the position on its circle of

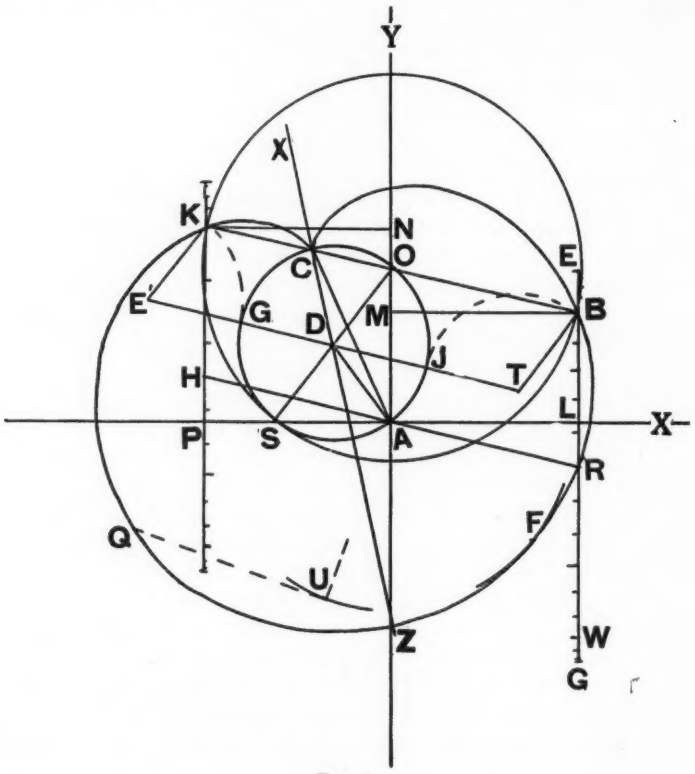


FIG. 1.

the cusp C of the cardioid that will be generated, since this cusp C must lie on the diameter through O and B. The same cardioid is drawn when β is increased or diminished by 180°, and the pen set down at K instead of at B. Measured from their common point of tangency S on their respective circles in a positive or anti-clockwise direction the distances of the cusp C and the starting point B or K are equal, that is, arc $SAJOC$ = arc SB, and the first increased by a whole circumference is equal to the arc SBK, so that as β grows from 0° to 360° the cusp C swings round its circle twice with uniform speed. The axis CZ of the cardioid will of course pass through D.

From what has just been said about α and β, it follows that two or more pens may be set down at various points on the starting circle, fastened together, and made to move with the same simple harmonic mechanism: they will all trace equal cardioids which have their cusps on the same small cusp circle; and a pair of diametrically opposite pens will trace the same cardioid.

Other Initial Phases and Starting Angles.—Various figures are here given to illustrate the changes that come upon the position of the cardioid as α and β are given different values, the cardioid however being always of the same size as long as the amplitude of the simple harmonic motion remains the same. We will at this juncture call attention only to Figs. 1, 2, 3. Fig. 1, as said before,

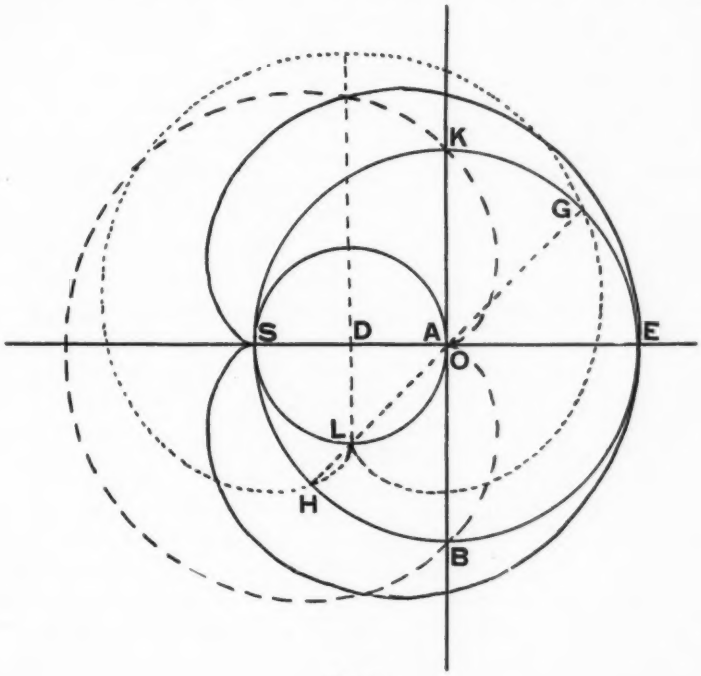


FIG. 2.

gives the values of $\alpha = 52°$ and $\beta = 77°$ or $257°$. In Fig. 2, we have $\alpha = 0°$, when the center O of the starting circle is at A, the center of the disk; the center D of the cusp circle is on the axis of X to the left of A, and the point of tangency of both circles is at S. When $\beta = 0°$ or $180°$, the pen is set down at B or K, and the cusp of the cardioid is at A. When $\beta = 90°$ or $270°$, the pen is started at E or S, and the cusp is at S. The cusp is always at the starting point when this is at the point of tangency or $180°$ away from it. When $\beta = 135°$ or $315°$, the pen is set down at G or H, and the cusp is at L. It will be seen that the cusp is always on the diameter through O and the starting point.

When $0° < \alpha < 90°$ we have a drawing like Fig. 1.

When $\alpha = 90°$ as in Fig. 3, the center O of the starting circle is at the distance unity from A, and the center D of the cusp circle is halfway between them, their point of tangency being at A. When $\beta = 0°$ or $180°$, the pen is set down at A or K. When $\beta = 35°$ or $215°$, the point is B or F and the cusp is at C.

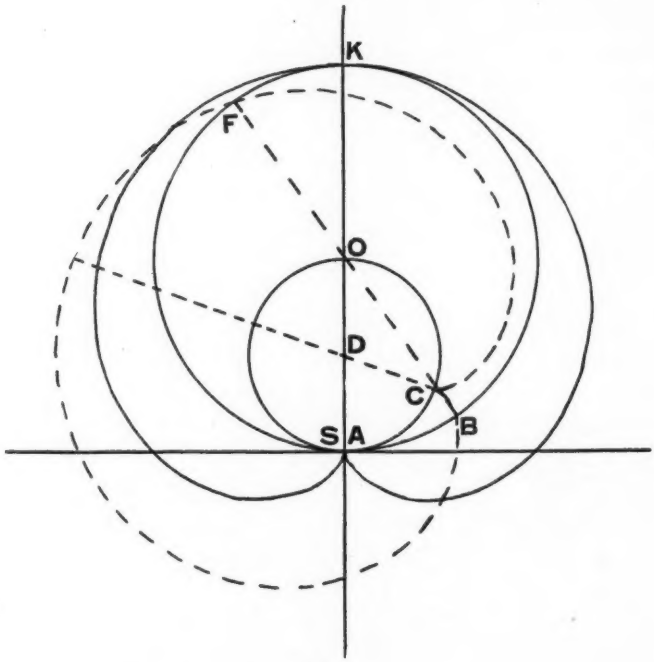


FIG. 3.

The easiest way to draw a cardioid is to take $\alpha = 90°$ and $\beta = 0°$, that is, to place the pen exactly at the center of the disk A when its phase is 90°. This is the way in which the writer had drawn a number of cardioids, when it occurred to him to displace the pen from A to some other point such as B, to see what would happen. Fortunately B was on the starting circle, and to his great surprise a perfect cardioid resulted. But the position of B had not been marked, so that in endeavoring to rediscover it later, hours of experimentation were necessary before its position was determined to be on the starting circle. This was changing β. Then the idea came to change the initial phase α. After much more experiment and analysis, the final law was found which is presented in this article.

When $90° < \alpha < 180°$, we have a diagram like Fig. 1 when this is turned half-way round the Y axis and looked at from the other side of the paper, the points D and S then being to the right of OA.

When $\alpha = 180°$, we have Fig. 2 when the paper is turned half-way round the point A with the same side of the page facing us. For the values of α from 180° to 360°, the points O and D are below the X axis, and the figures are obvious enough after a little study.

Object of this Article.—The chief object of this article is to prove that under the given conditions the curve so drawn is a cardioid.

The method to be pursued in the proof will be to suppose the cardioid generated as an epicycloid, and then to show that the pen traces the same curve. Taking the cusp circle, then, as the fixed one, the cusp C must evidently lie upon it and the axis of the cardioid must pass through its center. Placing the equal generating circle with its center at T in Fig. 1, when its tracing point is at B, we have the arcs COJ and BJ equal, and the lines CB and DT parallel. As the quadrilateral $DOBT$ has its opposite sides equal, they are also parallel, and hence OB is likewise parallel to DT, so that the lines CB and OB are coincident, and therefore the cusp C lies on the diameter KOB.

The Phase of the Pen at the Cusp.—For the sake of greater clearness we will suppose the disk to remain at rest and the pen to have its rotary motion in the opposite sense, that is, in an anti-clockwise direction. We will also assume for the present that the pen really traces the epicycloidal cardioid drawn by revolving the generating circle T with its pen at B about the fixed circle D. This assumption will be given a rigorous proof later, but it will serve well now to establish some of the conditions, such as the position of the cusp C and the phase of the pen when at the cusp.

It is evident that in passing through the cusp C the pen must be momentarily at rest on the paper and its rectilinear and rotary velocities must be equal and opposite. The diagram offers the suggestion that the pen is then at one of its two least distances from the center of revolution A, and that AC must be equal to $AL = MB = \sin \beta$. As there are two points on the cusp circle at this distance from A, we must take the one that is on the diameter KOB, as said before.

We must next find the two phases of the pen in its rectilinear motion when it is at L, at its minimum distance from A. As R is its mean position at the phase 0°, LR is equal to the sine of the phase when at L. Now as AO is equal and parallel to BR, OB and AR must also be equal to one another and parallel so that the triangles BOM and ARL are equal, and then $LR = OM = \cos \beta = \sin (90 \pm \beta)$. The phase of the pen when at the cusp is then either $90° - \beta = 90° - 77° = 13°$, or $90° + \beta = 90° + 77° = 167°$. In this instance we must take the latter value, because the direction of the linear motion of the pen must be downward at L in order to be to the right at C and annul its rotary motion which is counter-clockwise or to the left at C.

The rectilinear speed of the pen at L, or at C, is then $d[\sin (90° + \beta)] = \cos (90° + \beta) d\beta = -\sin \beta d\beta$. The rotary velocity at C is $CAd\theta = +\sin \beta d\theta = +\sin \beta d\beta$, because $d\theta$, the angular velocity of the pen, $d\beta$ or $d\alpha$, the variation of the rectilinear phase of the pen, are all equal and constant. Hence as the rectilinear velocity of the pen is $-\sin \beta d\beta$ and its rotary speed is $+\sin \beta d\beta$, it is at

rest at the cusp C and is then in the phase $90° + \beta$. The pen is in phase $90° - \beta = 13°$, in the present instance, when at F, where the rectilinear and rotary motions are also equal but in the same direction. There can be no other points besides C and F at which the two motions of the pen are exactly equal and either in directly opposite or in the same direction, because as the rotary motion must be at right angles to the radius through A, the two L points of the rectilinear motion are the only ones in which this is true.

As CA in the cusp circle is equal to $\sin \beta$, it follows that the angle $CDA = 2\beta$, and $DCA = DAC = 90° - \beta$. The angle $DSA = \alpha$, since $OA = \sin \alpha$ and SO = unity, and hence also $SA = \cos \alpha$, and $DAS = OCA = \alpha$. The position of the axis of the cardioid is therefore readily found. It is only by accident however that the cardioid passes through R, and that Z seems to lie on the Y axis.

The Second Starting Point K.—On account of the equality of the angles SDJ, SOB, DTB, CDJ, the (reflex) angle CDS is double the angle SOB, so that if the angular positions of the pen at B and of the cusp C are reckoned from the common point of tangency S of their circles, the angle of the cusp is twice that of the initial position of the pen and the arcs $SAJOC$ and SB are equal. If we add equals to these equals, the whole circumference of the D circle to the first and a semicircumference of the O circle to the second, it follows that the pen may be set down at K as well as at B in order to trace the same cardioid. The explanation given before for the point B then applies equally to the point K, if we replace LR by HP, BR by HK, and remember that as β is then increased by 180° or $KOA = 180° - \beta$, NO and HP are equal to $-\cos \beta = -\sin (90° \pm \beta) = \sin (270° \pm \beta) = 347°$ or 193° in the present case. The first of these $270° + \beta = 347°$ must evidently be here the phase of the K pen when at the cusp C, because the rectilinear motion at P must be upward, or to the right at C, to counteract the left rotary motion. As $90° + \beta$ and $270° + \beta$ differ 180°, the B and K pens are half a phase apart. And as their linear distance from one another is $KOB = 2 = (1 - \cos \theta) + (1 + \cos \theta)$, the line joining them passes through the cusp.

Preliminaries to the Proof that the Pen Traces the Cardioid.—We are now in a position to verify the assumption that the pen B traces the epicycloidal cardioid. First let us examine the rectilinear component of its motion. This would at any phase interval θ after passing through L, bring it, say, to W, so that its distance from L would be $LW = RW + RL = \sin [(90 + \beta) + \theta] - \cos \beta$. Here $90° + \beta$ is the phase of the pen at L, as we saw before, so that $(90° + \beta) + \theta$ is the phase after the interval θ. Taking $\theta = 135°$ as an example, this would make the phase $167° + 135° = 302°$, and put the pen at W, RW being equal to $\sin (90 + \beta + \theta)$. LW is in principle equal to the difference of RW and LR, that is, $RW - LR$ as would be evident if W were between L and R and both RW and LR positive. Here however, as RW is minus and LR plus, we have their numerical sum, which is minus. As $\sin ((90° + \beta) + \theta) = \sin (90° + (\beta + \theta)) = \cos (\beta + \theta)$, the length of $LW = \cos (\beta + \theta) - \cos \beta$.

Secondly, the rotary component alone of the pen after its arrival at the cusp

in the phase $90° + \beta$, during the phase interval θ after that, swings the pen from C to U through the angle $\theta = CAU$, which, as said, is here taken as 135°, the center of rotation being A, and the radius $UA = CA = AL = \sin \beta$.

Thirdly, in compounding the two motions of the pen, if we allow the rotary motion first to carry the pen from the cusp C to the point U, and then the rectilinear component to move it from U to Q over the tangent $UQ = LW = \cos(\beta + \theta) - \cos \beta$, we must, if our contention is correct, find the point Q on our cardioid with a position angle $QCX' = \theta$. The actual proof however will be the reversal of this procedure, taking a point Q on the cardioid at the position angle $X'CQ = \theta$, and showing that the length of the tangent QU dropped from it to the A circle is equal to $\cos(\beta + \theta) - \cos \beta$.

The Proof.—Taking C as the pole or origin and CX' as the positive direction of the X axis, the equation of the cardioid is $\rho = 1 - \cos \theta$, $SO = OB$ [$= \frac{1}{2} CZ$] being unity.

The coördinates of Q are then

$h = \rho \cos \theta = \cos \theta - \cos^2 \theta$;

$k = \rho \sin \theta = \sin \theta - \sin \theta \cos \theta$.

The radius of the A circle is $AU = AC = r = \sin \beta$.

The coördinates of its center are

$a = -AC \cos DCA = -\sin \beta \cos (90° - \beta) = -\sin^2 \beta$;

$b = -AC \sin DCA = -\sin \beta \sin (90° - \beta) = -\sin \beta \cos \beta$.

The square of the tangent QU dropped from a point (h, k) to the circle $(x - a)^2 + (y - b)^2 - r^2 = 0$ being $(h - a)^2 + (k - b)^2 - r^2$, we here have the square of QU equal to $(\cos \theta - \cos^2 \theta + \sin^2 \beta)^2 + (\sin \theta - \sin \theta \cos \theta + \sin \beta \cos \beta)^2 - \sin^2 \beta$.

Upon reducing, we find this value equal to the square of $\cos(\beta + \theta) - \cos \beta$. Therefore the pen B traces the cardioid $BCKQZFB$.

We may notice that the initial phase α does not appear in this analysis, because that merely fixes the positions of the centers of the starting and cusp circles O and D on the disk. The point B, however, with its starting angle β, being a point on the cardioid, determined the actual position of the cusp and the direction of the axis CZ. And lastly although definite numerical values of α, β, and θ, have been used in the figures for purposes of illustration, only general algebraic values have entered our equations.

Analysis and Synthesis of the Motions of the Pen.—The analysis just presented shows us how the cardioid is drawn mechanically. While the rotary motion swings the pen around the *tangent* circle (as we may call it) with uniform speed (see also Fig. 4) the rectilinear harmonic motion keeps it on the tangents to the successive points at the distance $\cos(\beta + \theta) - \cos \beta$. If we wished to plot these tangents as in Fig. 4, we would have to begin with a knowledge of the zero direction of the rotary motion. Its center is, of course, at A, and we know that the cusp C must be in the phase $90° + \beta$ ($= 167°$). Centering a protractor at A so that C reads $90° + \beta$, as in Fig. 4 in which the radii are drawn in dotted lines at intervals of 30°, we shall not only easily find the zero, but also see that L will indicate the initial phase α ($= 52°$), F will show $90° - \beta$ ($= 13°$), and D 180°.

As we wish to plot our tangents from the true 0° of the rotary motion, instead of from the cusp C from which we before reckoned the phase interval θ in our proof, we must subtract $90° + \beta$ from the θ there used. Changing back our $\cos(\beta + \theta)$ into $\sin(90° + \beta + \theta)$ from which it originated, and then subtracting $90° + \beta$ from θ, we have for the length of a tangent $\sin\theta - \cos\beta$, which, as $\cos\beta$ is constant, gives truly a rectilinear simple harmonic motion.

After the cardioid has actually been drawn, there is a very simple and expeditious way of finding at any time the phase of the pen at any point or conversely

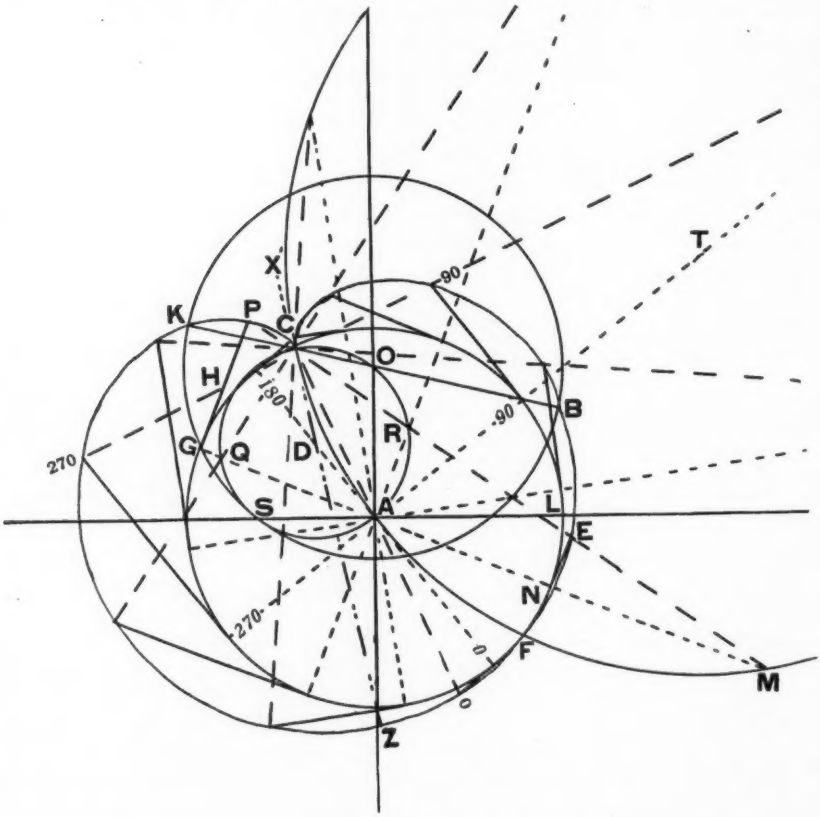


FIG. 4.

its position at a given phase. We need but center a protractor on the cusp C and take CA as the zero line. This has been done in Fig. 4, in which the radii vectores are drawn in dashed lines. The phases of the pen are then shown directly by the readings on the protractor. B will thus be found to show the initial phase α (= 52°), K will be $180° + \alpha$, X' $90° + \beta$ and Z $270° + \beta$. The points of the

cardioid at the phases 90° and 270° will be seen to have maximum, and those at $90 \pm \beta$ minimum distances from A.

While the mechanical method used in drawing the cardioid is really the tangent method just mentioned, the components of the movements, that is, the uniform angular speed of the pen about the center of the disk A, as well as its rectilinear simple harmonic motion, are bound to disappear individually from view on account of their combination. But if we refer now the motions of the pen to the cusp C, we find that the pen really moves about this cusp with uniform angular speed and also really moves in its direction with true simple harmonic motion, so that while the compound motion of the pen about the center of the disk is such as to make its components indistinguishable, these components, although united, are elegantly re-analyzed and exposed to view when we refer the motion to the cusp.

The Intersection Circle.—A study of Fig. 4 will reveal the peculiar property that the radii vectores passing through the cusp C and the radii of the tangent circle centered at A, that are in the same phase, intersect each other on what we may call the *intersection* circle which has its center at T. For example, the radius

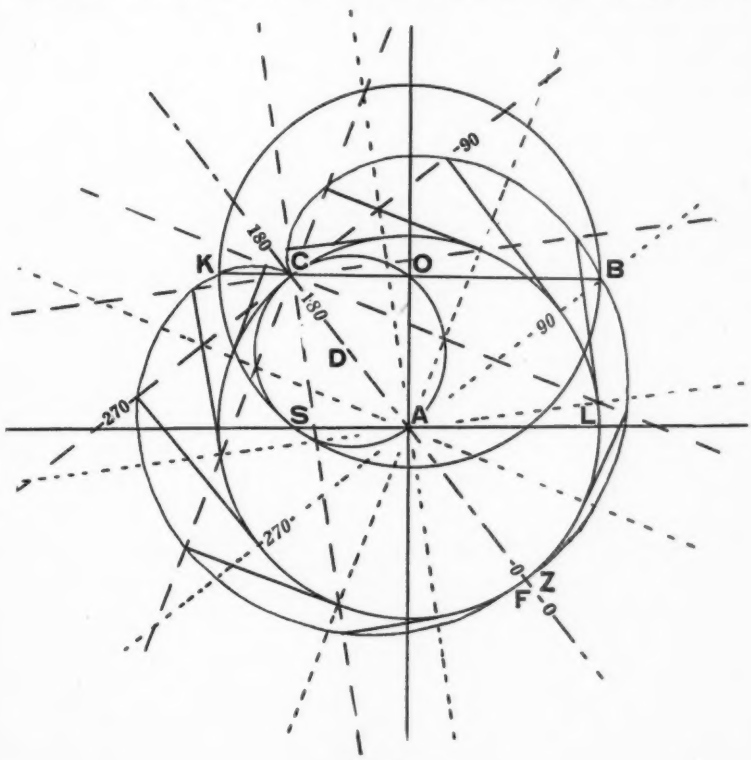


FIG. 5.

vector $PCREM$ of 30° (or 210°) and the radius $GQANM$ of the same phase, intersect in M which is on the intersection circle T. That this is really the case is due to the fact that the angle at M is constant and equal to $90° - \beta$. The 0° lines of the radii centered at C and A make this angle $CAD = 90° - \beta$, with one another, as we saw before in Fig. 1, so that all other radii in the same phase must also be inclined to one another at this angle, and hence the angle $PMG = 90° - \beta$. Therefore the intersection points of all the radii in equal phases lie on a circle, because they form triangles like MCA which has a constant base CA and a constant vertical angle M.

The length of the tangent NE (at 30°) is $+ \sin\theta - \cos\beta$ and of GP (180° away) is $- \sin\theta - \cos\beta$ or $\sin\theta + \cos\beta$, if we regard only its absolute value. Their difference $PH = 2\cos\beta$, and their distance apart $GN = 2\sin\beta$. Let us imagine the line HE drawn parallel to GN which has been omitted in order not to crowd the figure too much. Then we have the proportion

$$\frac{PH}{HE} = \frac{PG}{GA + AM} \quad \text{or} \quad \frac{2\cos\beta}{2\sin\beta} = \frac{\sin\theta + \cos\beta}{\sin\beta + AM},$$

from which we find the chord $AM = \sin\theta \tan\beta$. This chord becomes a maximum when $\theta = 90°$, so that the diameter of the intersection circle is $\tan\beta$ and its radius $AT = \frac{1}{2}\tan\beta$. Its center T lies on the 90° − 270° radius through A, produced if necessary, and its circumference passes through C, A and F.

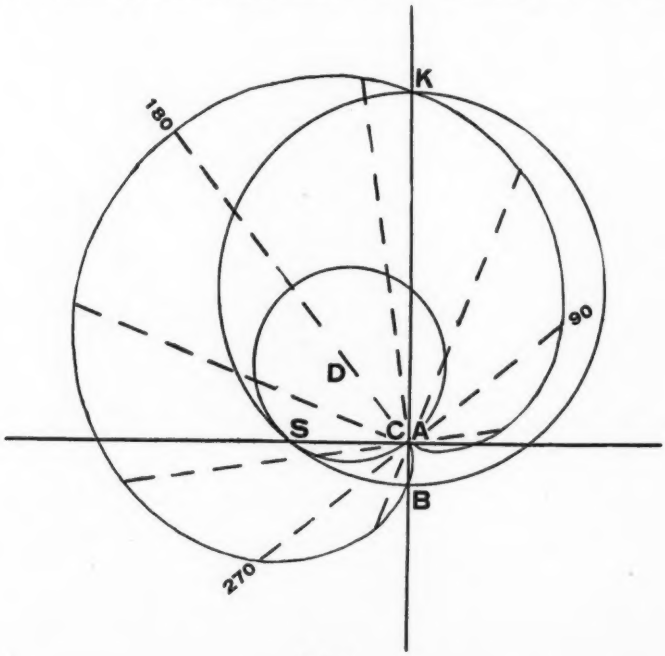


FIG. 6.

The points R and Q, at which a radius vector through C and a radius through A differing 90° in phase intersect, lie on the cusp circle, because the quadrilateral $CRAQ$ has two opposite angles at C and A equal to 90°, so that the other two are supplementary and are therefore inscribed in the circle passing through A and C.

In Fig. 5, α has its old value of 52°, but AOB or β has been made equal to 90°. The starting and cusp circles are in the same positions as in Fig. 1, because these depend upon α alone. The cusp C however, has shifted somewhat, so that it is now on the line DA since the angle $CAD = 90° - \beta = 0°$ in the present case. F is also on the line CDA, which is here the common 0° line of the radii passing through C and A. The radius AC or AL of the tangent circle is unity, and the lengths of the tangents $\sin\theta - \cos\beta$ drawn to it are now simply $\sin\theta$. The radius of the intersection circle $\frac{1}{2}\tan\beta$ is infinite, so that the radii through C and A at equal phases are parallel.

In Fig. 6, α is as usual 52°, but $\beta = 0°$. The starting and cusp circles and the 0° line DA are the same as before, but the cusp C is now at A. The radii of the tangent and intersection circles are zero. The lengths of the tangents $\sin\theta - \cos\beta$ are $\sin\theta - 1$ or $1 - \sin\theta$, and are ordinary radii vectores of the cardioid.

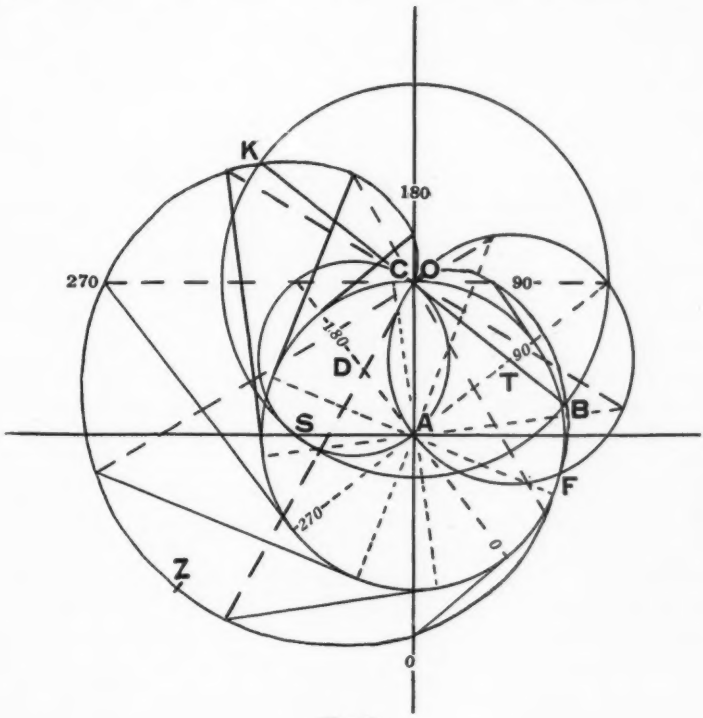


FIG. 7.

Finally in Fig. 7, $\alpha = \beta = 52°$. The starting and cusp circles and the 0° line of the tangent circle are the same as usual. The cusp C however is now at O, and the 0° line of the radii vectores through it is the negative direction of the Y axis. F has shifted somewhat and the intersection circle is much reduced.

COLLEGIATE MATHEMATICS FOR WAR SERVICE.

SEND COMMUNICATIONS FOR THIS DEPARTMENT TO H. BLUMBERG, University of Illinois.

STATISTICS IN RELATION TO THE WAR.

By ROXANA H. VIVIAN, Wellesley College.

Before 1914 there were many lines of work where the material dealt with was known to have the characteristics of statistical data, and to require treatment by one of the various statistical methods. Since 1914, not only are there many more fields where statistical methods are required because of their development in connection with war problems, but also, in the already established domains of statistics, the quantities to be collected and placed in systematic form for drawing conclusions have grown beyond the earlier proportions and require much more attention. Every new National Council or local committee needs immediate records of present conditions, and usually of the past as well. There is a growing interest in such statistical data and a desire for trained workers in large centers who can, and will, avoid expensive duplication in collecting, and who are able to present the authoritative abbreviated tables and charts that executive committees especially need.

This situation is withdrawing statistics from its position as a somewhat unpopular subject, and thoughtful workers and educators are making an effort to cancel the criticisms from which it has suffered because statistical problems were so often handled by inexperienced manipulators in ways that led to conclusions and plans that were disastrous in business and ludicrous in science. To schools and colleges this brings an opportunity for contributing certain preparation that is especially desired in the present emergency for the young men and young women who are being called to enter business, scientific, or government fields with the possibility of assuming some collection and supervision of statistical material, as well as the opportunity of giving to those who are going into different branches of military, naval or engineering service the underlying theory in those lines that the college class-room can so quickly furnish. For those who must present accurate tables of figures and draw far-reaching conclusions in the war problems of finance, research, national and international resources, the general background of college training is a substantial asset; and if to the general background there can be added even a brief elementary course in the mathematical foundations of statistics, a graduate is ready to enter more quickly into a responsible position and give reliable and necessary assistance.

This is not entirely a new point of view. For some years it has been evident that in many college subjects students need information in regard to statistical methods, as well as so-called statistics. Courses that deal with the general fundamental theory that is essential in any field have been introduced in various institutions. Except in a vocational institution where up-to-date files and computing machines are available, it is practically necessary to offer such work as a lecture course with special problems—that is, problems that require a minimum amount of time in collecting, tabulating, and making computations before drawing conclusions, and which yet afford opportunities for experimenting with all the known statistical methods and choosing the right one.

In attempting to present such work as a war emergency course, the situation differs very little from the problem of presenting the elements of statistical theory to a group of students whose college interests, or after-college plans, embrace all the natural sciences, psychology, education, economics, institutional accounting, commerical offices, municipal and governmental positions. The fundamental theory remains practically the same, the reading, required or suggested, can be slightly varied, and the problems can be chosen from the most essential fields of the present day, in the regions which are most open to the students. Such an arrangement is possible, although it may mean omission for the present of many of the interesting or vital sections of the subject, such as details of the history of government statistics in Belgium under Quételet, the development of insurance statistics beyond the possibility of business failure, the peculiar difficulties of early health statistics, and many more. The subject may still be presented after this adjustment as a combination course, which includes pure theory, supplemented by varied reading, with brief but ample work along especially suitable lines.

Books dealing with statistics can be found in many languages, but the standard collection for American students must be largely in English. During this year several new American books on statistics have been published and take their place among the older English and American volumes. Since they are all written from varying points of view, and since hardly one of the elementary books is sufficiently general, a reading shelf seems to be a prime necessity in beginning the study of statistics. In addition to this, careful references must be given to prevent a student from losing valuable time in securing the important first principles that are needed in all statistical work. Frequently a chapter will treat a simple collection by an advanced method, or a simple principle may be so veiled by logic, or by mathematics, as to be unrecognizable except by an expert in these lines. Part of the reading that can be done in such a collection becomes practice in extricating information, part furnishes the realization of how extremely varied and universal statistical problems and methods are, and the criticism of methods used by others leads often to the invention of new forms and short cuts by the student himself. Every one of these factors is an essential part of the mental resources of the future statistical clerk or statistician.

Daily, weekly, and monthly publications which students are reading today

with concern as to their accuracy and their predictions for the future, furnish problem material of deeper interest than many of the best and most convenient collections of preceding years. Questionnaires for registration show bewildering sources of difficulty in collecting material. Reports on daily receipts in Liberty Bond campaigns raise questions as to the complete reliability of published figures, and the necessity of checking results, and publishing the same with authorized signatures. Income tax classifications, national budget items, and appropriation lists with their large figures, are much more interesting to the average student and reveal problems which are more vital to him than the reports of corporations, organizations, and institutions, which were the usual forms of illustrating financial matters in statistics before the war. National vital statistics, and military and naval figures have put most of the scientific statistical data into the background; but since the methods remain the same the variation in material is relatively unimportant.

How such material shall be dealt with by students depends very much upon the recognized dividing line between academic and vocational courses. If adding machines are available, students can be taught to use them in a very short time, and more extensive problems can be undertaken without abnormal use of study hours for long clerical calculations. Unless such work includes obtaining information in regard to the mathematical principles of computation, and their application to machine computation, and unless the larger problems in collecting material and computing averages, percentages, deviations, and other figures are mentally worth the time they require, they lie outside the limits of a standard course for a college student. It is an interesting feature of the increasing demand for college courses in statistics that students will add, at an expense of their own time and money, an element of training which is vocational and not academic, namely, clerical experience with statistical collections once a week in some well-organized business office or municipal bureau where the exchange of collegiate theory for office experience seems to be a mutual and welcome courtesy.

The conclusion is rather a natural one that statistics with its steady general theory helping out the changing situation in these many fields is a worthy member of our mathematical family. The relation of the whole family to the war situation seems to be the customary one. It was jogging along as best it could and in the face of much unpopularity. Now the call comes to each member in this emergency; and statistics and all the other steeds are changing their harnesses and cutting across every field they can to reach the most strenuous and exacting positions and to carry the national problems to the speediest and best solution. Every institution and every instructor furnishing graduates who can assist in this particular drive is making a worthy contribution to the work of the war.

Note 1.—A few of the books advisable for a reading shelf for elementary statistical theory are the following:

Bailey & Cummings—Statistics.

Block—Traité Théorique et Pratique de Statistique.

Brinton—Graphic Methods.

Copeland—Business Statistics.
Elderton—Primer of Statistics.
King—Elements of Statistics.
Secrist—Introduction to Statistical Methods.
Thorndike—Mental and Social Measurements.
West—Introduction to Mathematical Statistics.
Yule—Introduction to the Theory of Statistics.

There are many that can be added along more general lines, and the *American Statistical Quarterly* and other magazines are furnishing an increasing number of articles on the statistical questions of the day.

Note 2.—Wellesley graduates who have had a one-hour per week course in statistical theory have secured positions in the Statistical Division of the Shipping Board and of the Labor Bureau. Their college work has covered, as completely as was possible in a short course, the collection and arrangement of statistical data, questionnaire forms, classification, graphic representation, computation of averages, percentages and deviations, methods of checking results, index figures, graphic comparison, and correlation.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

2737. Proposed by C. N. SCHMALL, New York City.

Employing Maclaurin's Theorem, or otherwise, expand the following three functions: (1) $e^{\tan^{-1}x}$ as far as x^6; (2) $e^{\sin x}$ as far as x^3; and (3) $\tan x$ as far as x^9.

2738. Proposed by W. D. CAIRNS, Oberlin College.

Prove that between any two points on a unit circle with its center at the origin there is a point whose coördinates are rational, *i. e.*, prove that the rational points on this circle are dense.

2739. Proposed by JOS. B. REYNOLDS, Lehigh University.

A particle is describing a smooth closed curve in a vertical plane. This curve is three feet high, $19\pi/4\sqrt{2}$ ft. long, and of such form that the pressure is always constant. Find the time of a complete revolution.

2740. Proposed by E. W. CHITTENDEN, University of Illinois.

Establish the identity (the determinant is of order n):

$$\begin{vmatrix} a-\lambda, & a, & a, & \cdots, & a \\ a, & a-\lambda, & a, & \cdots, & a \\ a, & a, & a-\lambda, & \cdots, & a \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a, & a, & a, & \cdots, & a-\lambda \end{vmatrix} = (-1)^{n-1}\lambda^{n-1}(na-\lambda).$$

2741. Proposed by HORACE L. OLSON, New Hampshire College.

Prove or disprove the following statement: If the three sides and the area of a triangle are integers, at least one of the three altitudes is an integer.

2742. Proposed by C. N. SCHMALL, New York City.

In Gregory's *Examples in the Differential and Integral Calculus*, 1841, Chap. VII, p. 124, ex. 22, I find the following celebrated problem: "To find a point within a triangle from which if lines be drawn to the angular points their sum may be the least possible." The author remarks that "the direct solution of this problem is long and complicated, etc."

Required a simple, brief solution.

2743. Proposed by H. S. UHLER, Yale University.

A certain compound pendulum (Borda) has the geometrical configuration resulting from the following assemblage of parts. The bob is a sphere supported directly by a rod of circular cross-section the axis of which coincides with an extended diameter of the sphere. (The rod does not project below the under surface of the sphere.) The upper end of the cylindrical rod is fitted perfectly in a hole in a rectangular block. The axis of the cylinder is parallel to the four vertical edges of the rectangular parallelepiped and it intersects the lower plane at the center of this rectangle. (The rod does not project above the upper surface of the block.) The knife-edges are formed by a right prism the lateral edges of which are horizontal and the cross section of which is an inverted isosceles triangle. The horizontal lateral face of the prism coincides with the upper plane of the rectangular block. The plane which contains the lowest lateral edge (axis of rotation) of the prism and bisects the associated diedral angle (median plane) also contains the (prolonged) axis of the rod. The triangular bases of the prism are parallel to, and equidistant from, the respective nearer vertical faces of the rectangular block. All parts are made of homogeneous invar and there are no cavities in the interior of the complete pendulum.

(*a*) Let $R \equiv$ radius of sphere,
$r \equiv$ radius of rod,
$h \equiv$ length of a vertical edge of the block (height),
$t \equiv$ length of a horizontal edge of the block which is parallel to the lateral edges of the prism (thickness),
$w \equiv$ length of a horizontal edge of the block which is perpendicular to the lateral edges of the prism (width),
$l \equiv$ perpendicular dropped from center of sphere upon the lowest lateral edge of the prism,
$a \equiv$ (vertical) altitude of isosceles section of prism,
$b \equiv$ (horizontal) base of isosceles section of prism,
$e \equiv$ total lateral edge of prism projecting beyond both sides of the block.

For extremely small oscillations of the frictionless pendulum the complete period P_0 may be represented by

$$P_0 = 2\pi\sqrt{\frac{N_1 + N_2 + N_3 + N_4 + N_5}{g(D_1 + D_2 + D_3 + D_4 + D_5)}}.$$

Prove that the symbols under the radical stand for the following expressions:

$$D_1 \equiv -\tfrac{1}{3}a^2be,$$

$$D_2 \equiv -\tfrac{1}{2}htw(2a - h),$$

$$D_3 \equiv \tfrac{1}{2}\pi r^2[(l - R)^2 - (a - h)^2],$$

$$D_4 \equiv \tfrac{4}{3}\pi lR^3,$$

$$D_5 \equiv \tfrac{1}{12}\pi k^2[4(l - R)(3R - 2k) + k(4R - 3k)],$$

$$N_1 \equiv \tfrac{1}{15}abe(12a^2 + b^2),$$

$$N_2 \equiv \tfrac{1}{12}htw(12a^2 - 12ah + 4h^2 + w^2),$$

$$N_3 \equiv \tfrac{1}{12}\pi r^2(a - h + l - R)[(a - h + l - R)^2 + 3(a - h - l + R)^2 + 3r^2],$$

$$N_4 \equiv \tfrac{4}{15}\pi R^3(5l^2 + 2R^2),$$

$$N_5 \equiv \tfrac{1}{60}\pi k^2[40kR^2 - 35k^2R + 4k^3 + 20(3R - 2k)(l - R)^2 + 10k(4R - 3k)(l - R)];$$

where $k \equiv R - \sqrt{R^2 - r^2}$.

(*b*) Let $x \equiv l - R$. If the pendulum beats seconds show that x must be a root of the cubic $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$, where

$$a_0 \equiv 80\pi^3r^3,$$

$$a_1 \equiv 40\pi[4\pi^2r^3 - 3r^2(g - 2\pi^2R) + 4\pi^2(R^2 - r^2)^{3/2}],$$

$$a_2 \equiv -20\pi\{4(g - 2\pi^2R)[2R^3 + 3r^2R + 2(R^2 - r^2)^{3/2}] + 3\pi^2r^4\},$$

$$\begin{aligned} a_3 \equiv & -[80\pi^3r^2(h - a)^3 - 120\pi gr^2(h - a)^2 + 60\pi^3r^4(h - a) - 5\pi^2abe(12a^2 + b^2) \\ & - 20\pi^2htw(12a^2 - 12ah + 4h^2 + w^2) - 80ga^2be + 120ghtw(h - 2a) - 224\pi^3R^5 \\ & + 160\pi gR^4 - 320\pi^3r^2R^3 + 240\pi gr^2R^2 + 60\pi^3r^4R - 60\pi gr^4 - 16\pi(14\pi^2R^2 + \pi^2r^2 \\ & - 10gR)(R^2 - r^2)^{3/2}]. \end{aligned}$$

(*c*) Given $a = 1$, $b = 0{\cdot}8$, $l = 2$, $g = 980{\cdot}5$, $h = 2$, $P_0 = 2$, $r = 0{\cdot}49/4$, $R = 12{\cdot}01/4$, $t = 0{\cdot}64$, and $w = 1{\cdot}2$; calculate l correctly to four, or more, decimal places.

(*d*) Compute the percentage error affecting the result if g were calculated from the true value of the period (2 secs), as obtained experimentally, while neglecting all moments of forces and of inertia except those pertaining to the spherical bob.

2744. Proposed by E. B. ESCOTT, Chicago, Ill.

An insurance company computes its quarterly premiums by adding 6 per cent. to the annual premium and dividing by 4. If a policyholder pays quarterly, what rate of interest is he paying?

2745. Proposed by G. I. HOPKINS, Manchester, N. H.

A recent English publication contains the following method of constructing a regular polygon of 17 sides: Draw the radius CB perpendicular to the diameter AQ of the circle whose center is B. On BC lay off BD equal to one-fourth of BC. On BA, lay off BE and draw DE making angle BDE one-fourth of angle BDA. On BQ lay off BF and draw DF making angle FDE 45°. On AF as diameter, construct semi-circle FRA intersecting CB in H. With E as center and EH as radius construct semi-circle LHK intersecting CB in H. At the points L and K draw the ordinates NL and MK. Bisect the arc NM and let P be the point of bisection. Then the chord $NP(= MP)$ is a side of the regular polygon of 17 sides. Is the method of construction correct?

SOLUTIONS OF PROBLEMS.

433 (Calculus). Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Solve the differential equation, $\dfrac{d^{1/2}y}{dx^{1/2}} - \dfrac{y}{x} = 0$.

DISCUSSION BY EMIL L. POST, Columbia University.

In the April number of the MONTHLY appeared two solutions of the fractional differential equation

$$\frac{d^{1/2}y}{dx^{1/2}} = \frac{y}{x}. \tag{1}$$

The first of these, $y = Cx^{-1/2}e^{-1/x}$, was obtained by reducing the given equation to an ordinary differential equation of the first order; while the second, $y = A_0(1 - i\sqrt{\pi}x^{-1/2} - 2x^{-1} + i\sqrt{\pi}x^{-3/2} + \ldots)$ was found by equating coefficients in an assumed expansion in series. Now each of the two methods used would seem to indicate that only one solution was possible, *i. e.*, the solution found by the corresponding method, yet the two solutions are clearly irreducible. A discussion of this difficulty might serve as a beginning of that more thorough discussion of the subject which the proposer of the problem desires.

Now first of all what does $d^{1/2}y/dx^{1/2}$ mean? More generally, what does $d^\mu y/dx^\mu$ mean where μ is any number? Before 1860, certainly, the method followed in answering that question was that due to Liouville. Starting from the known fact that $(d^\mu/dx^\mu)e^{ax} = a^\mu e^{ax}$, when μ was a positive integer, and also when μ was a negative integer, if by $d^{-p}\cdot y/dx^{-p}$ we mean the pth indefinite integral of y, he then assumed the relation to hold for all values of μ, and proceeded from that to obtain everything else. The most important formula that thus resulted was

$$\frac{d^\mu x^n}{dx^\mu} = (-1)^\mu \frac{\Gamma(-n+\mu)}{\Gamma(-n)} x^{n-\mu}. \tag{2}$$

It was this formula that the proposer of the problem used when he solved it by expansion in a series of powers of x.

But this method was too narrow, when compared with the general and rigorous methods of analysis then in use, to last. It was Riemann who first gave a definition of $d^\mu y/dx^\mu$ essentially like those used now. He gave it in the form of a definite integral. More recent writers express it as a contour integral. One of the most significant features of this newer development is the introduction of limits of "differentiation" thereby making fractional derivatives more like definite integrals than ordinary derivatives, while the ordinary derivative appears as a peculiar (though singularly important!) degeneration. The analogy of the general binomial expansion expressed as an infinite series, with its particular terminating form for a positive integral power is complete.

A useful form applicable only when the real part of the index of differentiation is negative is

$$\left\{\frac{d^{-\mu}}{dx^{-\mu}}\right\}_{x_0}^{X} f(x) = \frac{1}{\Gamma(\mu)}\int_{x_0}^{X}(X-x)^{\mu-1}f(x)dx. \tag{3}$$

It is to be noticed that when $x_0 = 0$, it becomes Riemann's form, while for $x_0 = -\infty$, it gives all the consistent results obtained by Liouville's method. As it stands it is more general than either.

The theorem that justifies the definition is that

$$\left\{\frac{d^{\mu_1}}{dx^{\mu_1}}\right\}_{x_0}^{X}\left\{\frac{d^{\mu_2}}{dy^{\mu_2}}\right\}_{x_0}^{X} f(y) = \left\{\frac{d^{\mu_1+\mu_2}}{dx^{\mu_1+\mu_2}}\right\}_{x_0}^{X} f(x), \tag{4}$$

where for positive integral indices we get the ordinary derivatives. The generalization of Leibnitz's theorem also follows, *i, e.*,

$$\left\{\frac{d^\mu}{dx^\mu}\right\}_{x_0}^{X} u\cdot v = u\left\{\frac{d^\mu}{dx^\mu}\right\}_{x_0}^{X} v + \mu\frac{du}{dx}\left\{\frac{d^{\mu-1}}{dx^{\mu-1}}\right\}_{x_0}^{X} v + \frac{m(\mu-1)}{2^3}\frac{d^2u}{dx^2}\left\{\frac{d^{\mu-2}}{dx^{\mu-2}}\right\}_{x_0}^{X} v + \cdots. \tag{5}$$

We are now in a position to reconcile the two solutions of equation (1). In reducing (1) to an equation of the first order, we used equation (5), and then equation (4) in the form

$$\frac{d^{1/2}}{dx^{1/2}}\frac{d^{1/2}y}{dx^{1/2}} = \frac{dy}{dx},$$

and

$$\frac{d^{-1/2}}{dx^{-1/2}}\frac{d^{1/2}y}{dx^{1/2}} = y.$$

Now although this procedure is valid when the lower limit, x_0, is finite, it is not valid when $x_0 = -\infty$, for we then have

$$\frac{d^{-1/2}}{dx^{-1/2}}\frac{d^{1/2}y}{dx^{1/2}} = y - B \tag{6}$$

where B is a constant depending on y.[1] The second solution can be now obtained in the same ways as the first.

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[1] The following explains the failure of the general theorem. From (3), when extended for all values, we find

$$\left\{\frac{d^\mu}{dx^\mu}\right\}_{x_0}^{X} B = B\frac{(X-x_0)^{-\mu}}{\Gamma(1-\mu)}.$$

If x_0 is not infinite on taking the μth derivative of this result, we get back C_1B. If x_0 is infinite, the above result is zero, the $-\mu$th derivative leaves it zero, and so the constant is lost, as (6) indicates.

Clearing (1) of fractions, operating through by $d^{1/2}/dx^{1/2}$ and using (5) we find

$$x\frac{d^{1/2}}{dx^{1/2}}\frac{d^{1/2}y}{dx^{1/2}}+\frac{1}{2}\frac{d^{-1/2}}{dx^{-1/2}}\frac{d^{1/2}y}{dx^{1/2}}=\frac{d^{1/2}y}{dx^{1/2}}$$

Using (4), (6), and (1) this reduces to

$$x\frac{dy}{dx}+\tfrac{1}{2}(y-B)=\frac{y}{x}$$

whose solution is

$$y=x^{-1/2}e^{-1/x}\left[\frac{B}{2}\int x^{-1/2}e^{1/x}dx+C\right]. \tag{7}$$

Since the order of the equation solved was raised, the best we can say is that if a solution of (1) exists for $x_0=-\infty$ it is contained in (7). As a check, we expand (7) in series obtaining

$$y=B+Cx^{-1/2}-2Bx^{-1}-Cx^{-3/2}+\cdots,$$

which, on comparison with the known solution for $x_0=-\infty$, gives $B=A_0$; $C=-i\sqrt{\pi}A_0$. Since C is no longer arbitrary, we must change the indefinite integral of equation (7) to a definite integral. We then find,

$$y=A_0\left[1-i\sqrt{\pi}x^{-1/2}e^{-1/x}+x^{-1/2}e^{-1/x}\int_{-\infty}^{x}t^{-3/2}e^{1/t}dt\right], \tag{8}$$

which is valid for all except positive real values of x.

The derivation of the solution

$$y=Cx^{-1/2}e^{-1/x}, \tag{9}$$

which was obtained in the same manner as (7) except that $B=0$, was not rigorous since it was not checked. If equation (1) be written

$$y=\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{y}{x}\right)$$

we can check directly by the use of (3). In fact

$$\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{y}{x}\right)=\frac{c}{\Gamma(\frac{1}{2})}\int_{x_0}^{X}(X-x)^{-1/2}x^{-3/2}e^{-1/x}dx=\frac{c}{\Gamma(\frac{1}{2})}X^{-1/2}e^{-1/X}\int_{0}^{(1/x_0-1/X)}t^{-1/2}e^{-t}dt$$

which is obtained by letting $x=X/(1+tX)$.

Since $\int_0^\infty t^{-1/2}e^{-t}dt=\Gamma(\frac{1}{2})$, if we let $x_0=0$ we find

$$\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{y}{x}\right)=CX^{-1/2}e^{-1/X}=y$$

which checks the solution. Clearly $\{d^{-1/2}/dx^{-1/2}\}_0{}^X(y/x)$ exists only when the real part of X is positive. The two solutions thus supplement each other.

Suppose now that $x_0\neq 0$, and yet is finite. Is there no solution? Clearly we always have the trivial solution $y=0$. If however we restrict ourselves to real values of the variable and let $x_0<0$, the function $y=g(x)$ where $g(x)=0$ for $x_0\leq x\leq 0$, and $g(x)=cx^{-1/2}e^{-1/x}$ for $x>0$ clearly satisfies the auxiliary differential equation, and also the original one; since

$$\left\{\frac{d^{-1/2}}{dx^{-1/2}}\right\}_{x_0}^{x}\left(\frac{g(x)}{x}\right)=\left\{\frac{d^{-1/2}}{dx^{-1/2}}\right\}_{x_0}^{x}cx^{-3/2}e^{-1/x}=g(x);\ x>0,$$

$$\left\{\frac{d^{-1/2}}{dx^{-1/2}}\right\}_{x_0}^{x}\left(\frac{g(x)}{x}\right)=0=g(x);\ x\leq 0.$$

However, (8) and (9) are the only analytic solutions here found. It is noteworthy that they correspond to the definition of the "generalized derivative" given by Liouville and Riemann respectively.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

AMERICAN MATHEMATICIANS IN WAR SERVICE

The following list includes the known names of mathematicians of the United States in national service occasioned by the war. So far as possible, the name of the institution from which each entered service, or the home address, and the branch of the service (including Y. M. C. A. and other non-combatant branches) are named. It is not ordinarily feasible to attempt to give present addresses. Corrections and additions will be gladly received by the Secretary-Treasurer of the Mathematical Association of America, W. D. Cairns, Oberlin, Ohio. A few known names from Canada are included in the list.

O. S. ADAMS, Coast and Geod. Survey; 1st Lieut. under War Dept.
L. K. ADKINS, La Crosse St. Normal; 1st Lieut., Field Art.
J. W. ALEXANDER, Princeton Univ.; Ord. Dept. with Major Moulton.
C. S. ALLIN, Queen's Univ.; Can. Engineers.
N. H. ANNING, Chilliwack (B. C.) Pub. Schools; Sgt., Can. Rwy. Troops.
J. J. ARNAUD, Coll. of the City of New York. Aberdeen Provg. Grd.
C. S. ATCHISON, Washington & Jefferson Coll. Auditor and cost accountant, Emergency Shipping Bd.
R. W. BARNARD, Univ. of Michigan; Instr. in math., F.A.C.O.T.S., Camp Taylor.
D. F. BARROW, Yale Univ.; National service.
RALPH BEATLEY, Horace Mann Sch.; 2nd Lieut., Coast Art.
D. R. BELCHER, Adelbert Coll.; Corp., N. A., in France.
A. A. BENNETT, Univ. of Texas; Capt., Ord. R. C.
W. J. BERRY, Brooklyn Polytech. Inst.; 2nd Lieut., 308th Inf., N. A.
HERMAN BETZ, Univ. of Michigan; Ord. Dept., N. A.
E. G. BILL, Dartmouth Coll.; Milit. Service Branch, Dept. of Justice, Ottawa.
R. L. BLANCHARD, Brown Univ.; 1st Lieut., Art., in France.
H. F. BLICHFELDT, Stanford Univ.; Aberdeen Provg. Grd.
G. A. BLISS, Univ. of Chicago; Aberdeen Provg. Grd.
PIERRE BOUTROUX, Princeton Univ.; French nat'l service.
J. B. BRANDEBERRY, Toledo Univ.; Naval Aux. Reserve.
J. N. BROADLICK, High Sch., Pittsburgh, Kan.; Battery, 130th F. A.
B. H. BROWN, Grad. Sch. Brown Univ.; Corp., 1 Cl., M.D.
THOMAS BUCK, Univ. of Calif.; 1st Lieut., Ord. Dept. with Major Moulton.
R. W. BURGESS, Brown Univ.; Bureau of Statistics, Washington, D. C.
H. T. BURGESS, Univ. of Wisconsin; Forest Products Laboratory, U. S. Dept. of Forestry.
C. C. CAMP, Ottawa Univ.; Sgt., Chem. Warfare Serv., A.E.F.
A. D. CAMPBELL, Northwestern Univ.; Lieut., N.A.
J. W. CAMPBELL, Wesley Coll. Winnipeg; Art., overseas.

E. F. CANADAY, Columbia, Mo.; Co. B, 342 M.G.B.N.
J. A. CAPARÓ, Univ. of Notre Dame; U. S. Air Service Sch. of Radio Mechanics.
H. L. CARD, Lombard Coll.; Sgt., Co. C, 1st Regt., Engrs., A.E.F.
F. L. CARMICHAEL, Grad. Sch., Princeton Univ.; 2nd Lieut., Aberdeen Provg. Grd.
MARY E. CASTER, Paterson, N. J.; Teaching blind soldiers, France.
C. W. COBB, Amherst Coll.; Instr., Govt. Aviation Service.
J. F. CONNER, Catholic Univ. of America; 1st Lieut., Bur. of Supplies and Accts., U. S. Navy.
J. L. COOLIDGE, Harvard Univ.; Major, Ord. Dept. heading scientific mission to Europe.
D. H. CRESSY, U. S. Milit. Acad.; Commandant, govt. sch. of aëronautics, Princeton.
W. L. CRUM, Yale Univ.; N.A.
C. H. CURRIER, Brown Univ.; Lecturer on Astr., Y. M. C. A., Summer, 1918
J. E. DAVIS, Pa. State Coll.; 313th Inf., N.A.
C. R. DINES, Dartmouth Coll.; Aberdeen Provg. Grd.
THEODORE DOLL, Northwestern Univ.; 161st Depot Brigade.
H. R. DOUGHERTY, N. Y. Milit. Acad.; 1st Lieut., 5th Co., O.R.T.C.
LUCY T. DOUGHERTY, Kansas City (Kan.) High School; Red Cross, France.
ARNOLD DRESDEN, Univ. of Wisconsin; Y.M.C.A., France.
W. E. EDINGTON, Univ. of Illinois; Research Div., Signal Service.
D. S. ELLIS, Queen's Univ.; Maj., Can. Engineers.
C. A. EPPERSON, Kirksville (Mo.) Normal; 1st Lieut., C.A.O.R.C.
J. D. ESHLEMAN, Univ. of Rochester; Instr. in math. F.A.C.O.T.S., Camp Taylor.
H. J. ETTLINGER, Univ. of Texas; Instr., aviation ground sch.
G. C. EVANS, Rice Inst.; Capt. O., Special Mission, France.
. . . . FARRIS, Ala. Polytech. Inst.; O.R.T.C.
PETER FIELD, Univ. of Mich.; Major, Ord., Sandy Hook Provg. Grd.
L. R. FORD, Harvard Univ.; Instr. in charge of math., F.A.C.O.T.S., Camp Taylor.
C. H. FORSYTH, Dartmouth Coll.; Plattsburg coll. training camp, Summer, 1918.
P. A. FRALEIGH, Cornell Univ.; Ord., Aberdeen Provg. Grd.
H. D. FRARY, Univ. of Illinois; Dir., wood-testing plant for aëroplanes, Univ. of Wis.
PHILIP FRANKLIN, Coll. of the City of New York; Aberdeen Provg. Grd.
T. C. FRY, New York City; Industrial Research, War Problems.
HAIG GALAJIKIAN, Princeton Univ. Aberdeen Provg. Grd.
H. M. GEHMAN; Aberdeen Provg. Grd.
B. P. GILL, Coll. of the City of New York; Aberdeen Provg. Grd.
R. E. GILMAN, Cornell Univ.; Capt., Coast Art.
S. C. GODFREY, U. S. Milit. Acad.; Maj., Engineers.
C. F. GREEN, Univ. of Ill.; Aviation, France.

T. H. GRONWALL, New York, N. Y.; Aberdeen Provg. Grd.
D. J. GUY, Whitworth Coll.; Govt. Serv., Washington, D. C.
P. W. HARNLY, Campbell Coll.; Natl. Service.
W. L. HART, Harvard Univ.; Lieut., Coast Art.
A. S. HAWKESWORTH, Sheridanville, Pa.; Mathematician, Ord., Navy Dept.
W. E. HEAL, Washington, D. C.; U. S. Bur. of Plant Industries.
C. M. HEBBERT, Univ. of Ill.; Instr., School of aviation.
M. HEDLUND, Beloit Coll.; 2nd Lieut., Adj. Gen's. Off.
T. H. HILDEBRANDT, Univ. of Mich.; Instr. in math., F.A.C.O.T.S., Camp Taylor.
L. S. HILL, Princeton Univ.; Ensign, U. S. Navy.
P. W. HILL, Depauw Univ.; 42nd Div., Field Art., A.E.F.
T. R. HOLLCROFT, Columbia Univ.; Instr. in math., F.A.C.O.T.S., Camp Taylor.
J. M. HOWIE, Peru (Neb.) St. Normal; Y.M.C.A., Camp Dodge.
E. P. HUBBLE, Grad. School Univ. of Chicago; Capt. O.R.C.
E. V. HUNTINGTON, Harvard Univ.; Maj. N. A. assigned to chief of staff, Statistical Branch, War Dept.
DUNHAM JACKSON, Harvard Univ.; Capt. Ord., with Major Moulton.
RALPH KEFFER, Grad. Sch. Harvard Univ.; Aviation.
O. D. KELLOGG, Univ. of Mo.; U. S. Nav. Exper. Sta., New London, Conn.
L. M. KELLS, Univ. of Ill.; O.R.T.C.
A. M. KENYON, Purdue Univ.; Acctg. Dept. Purdue cantonment, summer 1918.
S. D. KILLAM, Univ. of Alberta; Can. Art.
EDWARD KIRCHER, Harvard Univ.; Capt., Coast Art.
W. D. LAMBERT, Coast and Geod. Surv.; 1st Lieut. under War Dept.
W. W. LANDIS, Dickinson Coll.; Y.M.C.A., Italy.
A. O. LEUSCHNER, Univ. of Calif.; Supervisor of instruction, navig. schools on Pacific Coast, U. S. Shipping Board.
W. R. LONGLEY, Yale Univ.; Ballistic Div., Dupont Powder Co.
W. D. MACMILLAN, Univ. of Chicago; Major Ord., with Major Moulton.
WILLIAM MARSHALL, Purdue Univ.; Controller and chief statistician, Internat. Sugar. Comm.
MAX MASON, Univ. of Wisconsin; Submarine research.
J. V. MCKELVEY, Cornell Univ.; 2nd Lieut., 312th Inf. N.A.
N. B. MACLEAN, Univ. of Manitoba; Major Royal Garrison Artillery, France.
A. S. MERRILL; Naval statistical work, Philadelphia (?).
W. V. METCALF, Oberlin, Ohio; Red Cross, France.
J. S. MIKESH, Hibbing (Minn.) Jr. Coll.; Aberdeen Provg. Grd.
C. W. MILLER, Harvard Univ.; Gov't inspector of optics.
W. E. MILNE, Bowdoin Coll.; 1st Lieut., Ord. R.C.
H. H. MITCHELL, Univ. of Pa.; Aberdeen Provg. Grd.
G. R. MIRICK, New Castle (Pa.) High Sch.; In charge of benzol and toluol dept. of a coke co. for the government.
E. H. MOORE, Univ. of Chicago; Chairman representing mathematics, Natl. Research Council.

H. C. M. MORSE, Harvard Univ.; Ambulance Corps, France.
F. R. MOULTON, Univ. of Chicago; Major Ord. R.C.
J. R. MUSSELMAN, Univ. of Illinois; 1st Lieut., Statistic. Staff, Food Commission.
J. J. NASSAU, Syracuse Univ.; 303 Regt., Engrs., in France.
I. I. NELSON, Austin (Texas) High Sch.; N.A.
A. H. NORTON, Elmira Coll.; Y.M.C.A. in France.
J. A. NYSWANDER, Univ. of Nevada; N.A.
E. J. OGLESBY, William and Mary Coll. Capt., instr. in gunnery, orientation and material, Coast Art. Tr. Camp, Fort Monroe.
——— OWENS, Ala. Polytech. Inst.; O.R.T.C.
T. A. PIERCE, Harvard Univ.; Lieut. U.S.R.
FLORA PORTER, Nashville, Tenn.; Index Clerk, War Dept.
H. E. PORTER, Kansas St. Agric. Coll.; N.A.
P. C. PORTER,, Rusk (Texas) Acad.; Aviation.
F. D. POSEY, Lebec, Cal.; 2nd Lieut., 21st Inf., N.A.
H. H. PRIDE, New York Univ.; 2nd Lieut., N.A.
W. R. RANSOM, Tufts Coll.; Summer navig. schools, U. S. Shipping Board.
F. W. REED, Univ. of Illinois; Instr. School of aviation.
L. J. REED, Univ. of Maine; Statistician, War Trade Board.
A. L. RHOADES, Fort Monroe; Lieut-Col., in France.
J. N. RICE, Catholic Univ. of Amer.; N.A.
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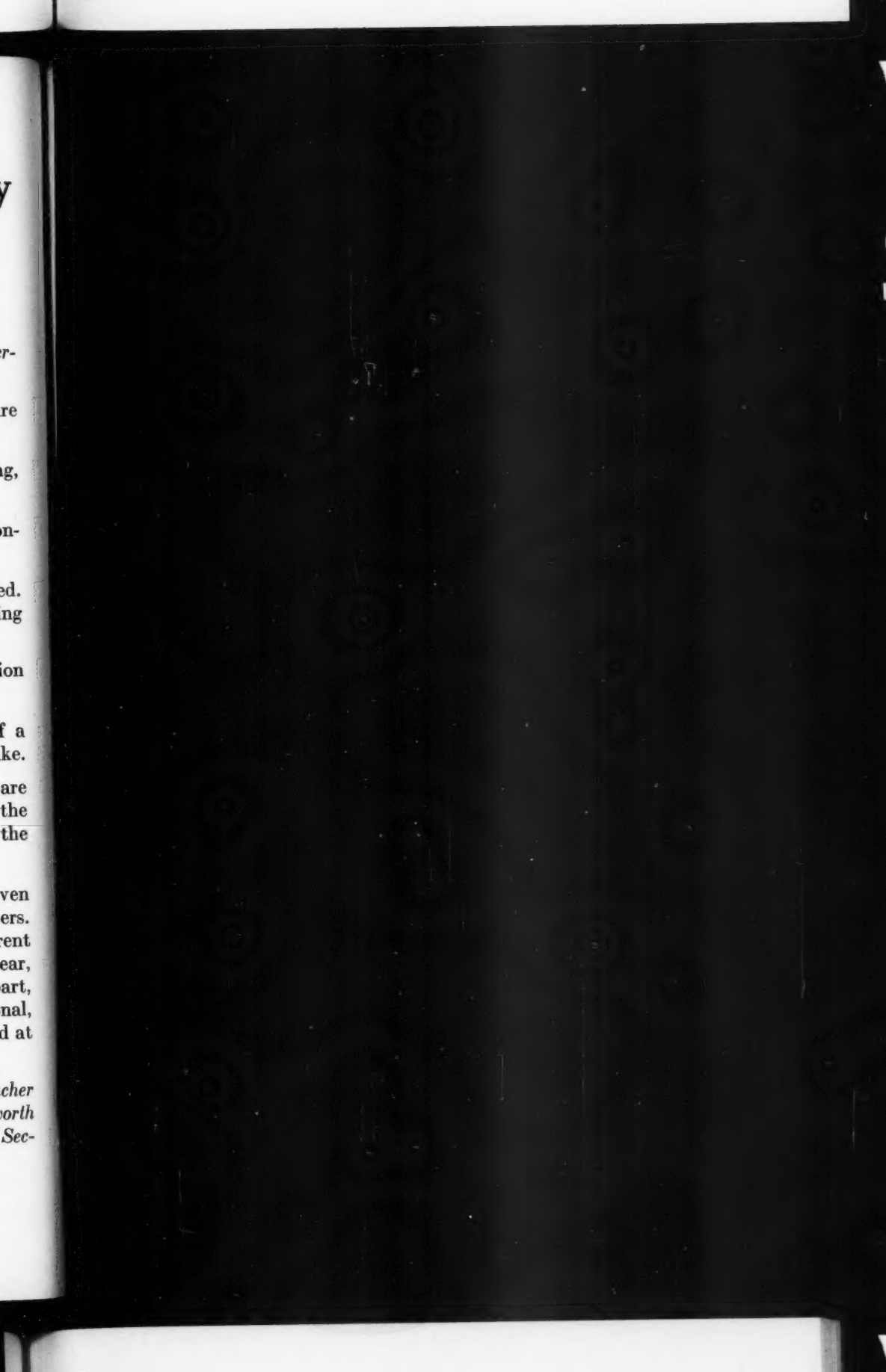

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ON CERTAIN CONSTRUCTIONS OF DESCRIPTIVE GEOMETRY.

EXTRACTS FROM LETTERS ADDRESSED BY

GINO LORIA, University of Genoa, Italy,
to W. H. ROEVER, Washington University, St. Louis.

I.

It will doubtless not have escaped you that the month of July of this year marks the first centenary of the death of Gaspard Monge. The war has rendered impossible the execution of my project to publish at this time, as a monument to the memory of this immortal savant, my *History of Descriptive Geometry*, finished four years ago and intended to form the third volume of my *Vorlesungen über darstellende Geometrie*. Nevertheless my mind has recently been occupied —perhaps because of this centenary—with a group of questions, bearing on developments which have been influenced (or are likely to be influenced) by the ideas of this great mathematician, concerning that branch of geometry in which you are especially interested. Will you permit me therefore to share with you some of my thoughts on this subject?

Descriptive geometry had birth in the mind of Monge as an auxiliary to the engineer, destined (as he says in the beginning of his work) "to draw the French nation away from the dependence which it has had up to this time on foreign industry." He concerned himself then with a science, which certainly in an embryonic form, is found among all civilized peoples. But while architects and painters desire in general to represent objects as they appear, Monge has given a process for performing (by drawings in a plane) all sorts of geometric operations on figures of three dimensions. Hence one may say, that, while the method of double (orthographic) projection as used, for example, by Vitruvius in his celebrated work on architecture is *static* in character, with Monge it becomes